

# **Economics Department Discussion Papers Series**

ISSN 1473 – 3307

## **OPTIMISM AND PESSIMISM IN GAMES**

Jürgen Eichberger and David Kelsey

**Paper number 06/05**

# OPTIMISM AND PESSIMISM IN GAMES\*

Jürgen Eichberger  
Alfred Weber Institut,  
Universität Heidelberg, Germany.

David Kelsey  
Department of Economics,  
University of Exeter, England.

1st August 2005

## Abstract

This paper considers the impact of ambiguity in strategic situations. It extends the earlier literature by allowing for optimistic responses to ambiguity. Ambiguity is modelled by CEU preferences. We study comparative statics of changes in ambiguity-attitude in games with strategic complements or substitutes. This gives a precise statement of the impact of ambiguity on economic behaviour. We also consider the possibility that players may be overconfident in the sense of over-estimating the probability of favourable outcomes. This has a similar effect of increasing equilibrium strategies in games of strategic complements. Finally we consider RDEU preferences.

**Address for Correspondence** David Kelsey, Department of Economics, School of Business and Economics, University of Exeter, Rennes Drive, Exeter, Devon, EX4 4PU, ENGLAND.

**Keywords** Ambiguity in games, overconfidence, strategic complementarity, optimism, RDEU.

**JEL Classification** C72, D81.

---

\*Research supported by ESRC grant no. RES-000-22-0650. We would like to thank Dieter Balkenborg, Andrew Colman, Simon Grant, David de Meza, Frank Milne, Anna Stepanova, Jean-Marc Tallon and participants in seminars at the universities of Bielefeld, Birmingham, Exeter, FUR XI and RUD 2005 for comments and suggestions.

# 1 INTRODUCTION

This paper is concerned with uncertainty in strategic situations. We consider how alternative decision theories affect play in games with strategic complementarity. Although the paper focuses on ambiguity, we consider some other behavioural theories for comparison. In particular we consider overconfidence and Rank-Dependent Expected Utility (henceforth RDEU).

## 1.1 Ambiguity

The main focus of this paper is on ambiguity, which describes situations where individuals cannot or do not assign subjective probabilities to uncertain events. There is a substantial body of experimental evidence which suggests that people behave differently when probabilities are ambiguous see for instance Camerer and Weber (1992). The importance of the distinction between risk and ambiguity is confirmed by recent research, which shows that different parts of the brain process ambiguity and probabilistic risk, see Camerer, Lowenstein, and Prelec (2005). The majority of individuals respond by behaving cautiously when there is ambiguity. Henceforth we shall refer to such cautious behaviour as *pessimism*.<sup>1</sup> In experiments a minority of individuals behave in the opposite way which we shall refer to as *optimism*. (See for instance Camerer and Weber (1992) or Cohen, Jaffray, and Said (1985).) Moreover the same individual may be pessimistic in one situation and optimistic in another.

Models of the impact of ambiguity on individual decisions can be found in Gilboa and Schmeidler (1989), Sarin and Wakker (1992) or Schmeidler (1989). A theory of games with ambiguity has been proposed by Dow and Werlang (1994) and developed in Eichberger and Kelsey (2000). In Eichberger and Kelsey (2002) we presented some results on the comparative statics of ambiguity in games of strategic complements under the assumption of ambiguity-aversion.<sup>2</sup> Until now, this literature has usually assumed that players are uniformly averse to ambiguity.<sup>3</sup>

The present paper aims to extend the previous literature on ambiguity and strategic interaction by allowing for the possibility of optimism. We find that in games with strategic

---

<sup>1</sup>See Wakker (2001) who relates this pessimism to a generalised version of the Allais paradox.

<sup>2</sup>For some alternative approaches to modelling games with ambiguity see Lo (1999) and Marinacci (2000).

<sup>3</sup>One exception is Marinacci (2000), who assumes that players either display global ambiguity-aversion or global ambiguity-preference. However the evidence shows that the same individual can express ambiguity-preference in some situations and ambiguity-aversion in others. We consider such mixed ambiguity attitudes.

complements and positive externalities, an increase in optimism has the effect of increasing the equilibrium strategy. If a given player is optimistic (s)he places more weight on good outcomes than an expected utility maximiser would. In this case, good outcomes would be perceived to be situations where other players use high strategies. Strategic complementarity implies that over-weighting high strategies will increase the given player's incentive to play a higher strategy. In games of strategic substitutes the effect is reversed. An increase in ambiguity-aversion increases the weight placed on low strategy profiles of one's opponents and hence increases the marginal benefit of increasing one's own strategy. Consequently the strategies played in a symmetric equilibrium increase.

## 1.2 Overconfidence

There is a substantial body of psychological evidence, which suggests that individuals are over-confident in the sense that they overestimate the probability of good outcomes. In particular overconfidence seems to arise when the outcome depends on the individual's own performance rather than chance. A large majority of people consider that they are better than average drivers or their marriages are less likely than average to end in divorce, Svenson (1981), Baker and Emery (1993).

Many sub-optimal types of behaviour appear to arise from overconfidence. These include the dotcom boom, railway mania and real-estate bubbles. In de Meza (2004) it is argued that individuals are overconfident about their chances of success when setting up a new business. This has the implication that government policy to encourage business start-ups is misguided, since even in the absence of intervention there are too many start-ups.

There is a puzzle. Why does overconfidence persist given that it causes people to make sub-optimal decisions? A possible explanation is offered by Compte and Postlewaite (2001), who consider a situation where an individual's performance depends in part on his/her beliefs about the probability of success. In this case optimism may increase pay-offs, since although it causes sub-optimal decisions it also improves performance.

We believe that over-confidence may have greatest impact in strategic situations, since over-confidence by one individual may cause not just this individual to change his/her behaviour but also cause other individuals to change their behaviour. Hence overconfidence has the potential to give rise to multiplier effects. We find that when there is strategic complementarity,

over-confidence has the effect of increasing the equilibrium strategies. Consider a situation where a number of individuals are deciding how much to invest. Assume moreover that there is strategic complementarity between the investments of different individuals. It is plausible that over-confidence results in individuals over estimating both the total and marginal benefits of investment. Then overconfidence will induce any given individual to invest more than if (s)he had correct expectations. Strategic complementarity implies that other individuals will also want to invest more, which in turn will cause the first individual to increase investment. This may increase his/her pay-off and hence justify the original investment. This provides an alternative explanation of why overconfidence persists despite its apparent disadvantages. Overconfidence may give economic agents higher pay-offs in strategic interactions, since it commits them to play strategies which would not ordinarily be best responses but are nevertheless advantageous.

### 1.3 Rank Dependent Expected Utility

The third behavioural theory we study is rank-dependent expected utility (RDEU), see Quiggin (1982) and Yaari (1987). There is psychological evidence that individuals subjectively distort probabilities, see Gonzalez and Wu (1999). RDEU models this by assuming that individuals maximise a weighted sum of utilities. In this model there is an objective probability distribution. The weights are not probabilities but a transformation of probabilities based on the rank of the outcome. An example of this would be a very cautious person who always put more weight on the worst outcome than its probability would justify. RDEU is capable of explaining many of the anomalies, which have been observed in individual choice, such as the Allais paradox, the common ratio effect and the common consequences effect. It has been shown that this theory is superior to expected utility theory in explaining insurance purchase, Segal and Spivak (1990) and the demand for lottery tickets, Quiggin (1991).

Within the RDEU model, there are two possible sources of risk aversion. Diminishing marginal utility of wealth will produce risk-averse behaviour for the usual reasons. In addition risk-aversion may be produced by the distortion of probabilities. This is referred to as *probabilistic risk aversion*. In the present paper we study behaviour in games where the utility function is given. Thus the only source of changes in risk-aversion is from changes in the way probabilities are distorted. We consider a stochastic game where players have RDEU preferences. We show that if there is strategic complementarity, an increase in probabilistic risk

aversion will result in a decrease in equilibrium strategies.

## 1.4 Welfare Effects

If there are positive externalities between the players, then for the usual reasons, too few positive externalities will be produced in Nash equilibrium and hence it will be Pareto inefficient. Thus an increase in equilibrium strategies caused by an increase in overconfidence, an increase in ambiguity preference or a decrease in probabilistic risk-aversion may result in an ex-post Pareto improvement.

One may also consider an ex-ante measure of welfare. Consider the case of overconfident individuals. In this case the utility increase will be higher, since not only will higher strategies be played for any given state of nature but also overconfidence causes people to place more weight on more favourable states of nature. However since this effect is essentially based on an illusion, it does not seem correct to take it into account in a welfare analysis.

**Organisation of the Paper** In section 2 we present our framework and definitions. In section 3 we introduce our solution concept and prove existence of equilibrium. In section 4 we derive the comparative statics of changes of ambiguity attitude in games of strategic complements or substitutes. The impact of overconfidence in games is studied in section 5. Concluding comments can be found in Section 6. Appendix A relates a number of alternative notions of the support of a capacity and Appendix B contains the proofs of those results not proved in the text.

## 2 MODELLING AMBIGUITY IN GAMES

This section introduces Choquet Expected Utility (henceforth CEU), which is the main theory of ambiguity we shall use. First we shall present the framework and definitions.

### 2.1 Games with Aggregate Externalities

This paper focuses on ambiguity in games with aggregate externalities, (defined below). Consider a game  $\Gamma = \langle N; (S_i), (u_i) : 1 \leq i \leq n \rangle$  with finite pure strategy sets  $S_i$  for each player and payoff functions  $u_i(s_i, s_{-i})$ . Player  $i$  has a finite strategy set which, for convenience, we

identify with an interval of the integers,  $S_i = \{\underline{s}_i, \underline{s}_i + 1, \dots, \bar{s}_i\}$ , for  $i = 1, \dots, n$ .<sup>4</sup> The notation,  $s_{-i}$ , indicates a strategy combination for all players except  $i$ . The space of all strategy profiles for  $i$ 's opponents is denoted by  $S_{-i}$ . The space of all strategy profiles is denoted by  $S$ . Player  $i$  has utility function  $u_i : S \rightarrow \mathbb{R}$ , for  $i = 1, \dots, n$ .

**Definition 2.1** *A game,  $\Gamma$ , has positive (resp. negative) aggregate externalities if  $u_i(s_i, s_{-i}) = u_i(s_i, f_i(s_{-i}))$ , for  $1 \leq i \leq n$ , where  $u_i$  is increasing (resp. decreasing) in  $f_i$  and  $f_i : S_{-i} \rightarrow \mathbb{R}$  is increasing in all arguments.*

This is a separability assumption. It says that a player only cares about a one-dimensional aggregate of his/her opponents' strategies. An example would be a situation of team production, in which the utility of a given team member depends on his/her own labour input and the total input supplied by all other members of the team.<sup>5</sup>

A game is symmetric if all players have the same strategy set and pay-off function.

**Definition 2.2** *Let,  $\Gamma = \langle N; (S_i), (u_i) : 1 \leq i \leq n \rangle$  be a game with aggregate externalities. We say that  $\Gamma$  is symmetric if  $S_i = S_j, 1 \leq i, j \leq n; u_i = u, f_i = f$  for  $i = 1, \dots, n$ .*

**Notation 2.1** *Since  $S_{-i}$  is finite, we may enumerate the possible values of  $f_i$ ,  $f_i^0 < \dots < f_i^M$ . Since  $f$  is increasing  $f_i^0 = f(0, \dots, 0)$  and  $f_i^M = f(K_1, \dots, K_n)$ .*

Marginal benefit without ambiguity is defined in the usual way.

**Definition 2.3** *Let  $\Delta_i(s_i, s_{-i}) = u_i(s_i, s_{-i}) - u_i(s_i - 1, s_{-i})$ , i.e.  $\Delta_i(s_i, s_{-i})$  denotes the marginal benefit to individual  $i$  of increasing his/her action from  $s_i - 1$  to  $s_i$  when his/her opponents' strategy profile is  $s_{-i}$ . If  $f_i(s_{-i}) = f_i^r$ , we shall write  $\Delta_i(s_i, f_i^r)$  for  $\Delta_i(s_i, s_{-i})$ .*

Thus in a game of strategic complements (resp. substitutes) a given player's marginal benefit will increase (resp. decrease) when the aggregate of his/her opponents' strategy profile increases.

**Definition 2.4** *A game,  $\Gamma$ , with aggregate externalities is a game of strategic substitutes (resp. complements) if  $\Delta_i(s_i, f_i^r)$  is a strictly decreasing (resp. increasing) function of  $r$ , for  $1 \leq i \leq n$ .*

The following assumption will be a maintained hypothesis throughout sections 2 to 4.

**Assumption 2.1** *All games,  $\Gamma$ , are assumed to be concave, by which we mean that for all  $i$ ,  $u_i(s_i, s_{-i})$  is a strictly concave function of  $s_i$ .*

<sup>4</sup>It would be straightforward to extend the results to a multi-dimensional strategy space.

<sup>5</sup>For a more detailed analysis of the impact of ambiguity on team production see Kelsey and Spanjers (2004).

## 2.2 Non-Additive Beliefs and Choquet Integrals

First we introduce the CEU model of ambiguity, which represents beliefs as capacities. A capacity assigns non-additive weights to subsets of  $S_{-i}$ . Formally, they are defined as follows.

**Definition 2.5** A capacity on  $S_{-i}$  is a real-valued function  $\nu$  on the subsets of  $S_{-i}$  such that  $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$  and  $\nu(\emptyset) = 0$ ,  $\nu(S_{-i}) = 1$ .

Thus a capacity is like a subjective probability except that it may be non-additive. The simplest example of a capacity is the complete uncertainty capacity defined below.

**Definition 2.6** The complete uncertainty capacity,  $\nu_0$  on  $S_{-i}$  is defined by  $\nu_0(A) = 0$  for all  $A \subsetneq S_{-i}$ .

Intuitively  $\nu_0$  describes a situation where the decision maker knows which states are possible but has no further information about their likelihood.

**Definition 2.7** Let  $\nu$  be a capacity on  $S_{-i}$ . The dual capacity  $\tilde{\nu}$  is defined by  $\tilde{\nu}(A) = 1 - \nu(\neg A)$ .

The capacity and the dual capacity encode the same information. If beliefs are represented by a capacity  $\nu$  on  $S$ , the expected utility of the payoff obtained from a given act can be found using the Choquet integral, which is defined below.

**Notation 2.2** Let  $\Gamma$  be a symmetric game with aggregate externalities we shall use  $H_r$  to denote the event  $\{s_{-i} \in S_{-i} : f(s_{-i}) \geq f_r\}$ .

**Definition 2.8** Let  $\Gamma$  be a symmetric game with positive aggregate externalities. The Choquet integral of  $u_i(s_i, s_{-i})$  with respect to capacity  $\nu_i$  on  $S_{-i}$  is:

$$V_i(s_i) = \int u_i(s_i, s_{-i}) d\nu_i = u_i(s_i, f_M) \nu_i(H_M) + \sum_{r=0}^{M-1} u_i(s_i, f_r) [\nu_i(H_r) - \nu_i(H_{r+1})].$$

This is a special case of the Choquet integral, which applies in games of positive aggregate externalities. For the general definition see Schmeidler (1989).

**Definition 2.9** A capacity is said to be convex if  $\nu(A \cup B) \geq \nu(A) + \nu(B) - \nu(A \cap B)$ .



Schmeidler (1989) argues that convex capacities represent ambiguity-aversion.<sup>6</sup> More recently Wakker (2001) has argued that convexity is implied by a generalised version of the Allais paradox.

**Definition 2.10** *Let  $\nu$  be a capacity on  $S_{-i}$ . The core  $\mathcal{C}(\nu)$  is defined by*

$$\mathcal{C}(\nu) = \{p \in \Delta(S_{-i}); \forall A \subset S_{-i}, p(A) \geq \nu(A)\}.$$

The following result shows that for a convex capacity, the Choquet integral for a given act  $a$  is equal to the minimum over the core of the expected value over  $f$ . Hence convex capacities provide an attractive representation of pessimism. When a decision-maker does not know the true probabilities (s)he considers a set of probabilities to be possible and evaluates any given act by the least favourable of these probabilities.

**Proposition 2.1** *If  $\nu$  is an convex capacity on  $S_{-i}$ , then  $\int a d\nu = \min_{p \in \mathcal{C}(\nu)} \mathbf{E}_p a$ , where  $\mathbf{E}$  denotes the expected value of  $a$  with respect to the additive probability  $p$ .*

## 2.3 JP-Capacities

Next we introduce a class of capacities which are useful for representing the impact of ambiguity in games since they are capable of modelling both optimism and pessimism. Jaffray and Philippe (1997) consider the following class of capacities.

**Definition 2.11** *Say that a capacity  $\nu$  on  $S_{-i}$  is a JP-capacity if there exists a convex capacity  $\mu$  and  $\alpha \in [0, 1]$ , such that  $\nu = \alpha\mu + (1 - \alpha)\tilde{\mu}$ , where  $\tilde{\mu}$  denotes the dual capacity of  $\mu$ .*

This class of capacities allows preferences to be represented in both the multiple priors and CEU forms. Perceived ambiguity is represented by the capacity  $\mu$ , while ambiguity attitude is represented by  $\alpha$ . Hence JP capacities allow a distinction between ambiguity and ambiguity-attitude, which is formalised in the following definitions.

**Definition 2.12** *Let  $\nu$  and  $\nu'$  be two capacities on  $S$ . We say that  $\nu$  is more ambiguity-averse than  $\nu'$  if for all  $A \subseteq S$ ,  $\nu(A) \leq \nu'(A)$ .<sup>7</sup>*

---

<sup>6</sup>Alternative definitions of ambiguity-aversion have been proposed by Epstein (1999) and Ghirardato and Marinacci (2002).

<sup>7</sup>Theories of ambiguity-aversion have been proposed by Epstein (1999), Ghirardato and Marinacci (2002)

The following result shows that for JP-capacities, an increase in  $\alpha$  implies an increase in ambiguity-aversion.

**Proposition 2.2** *Suppose that  $\hat{\alpha} \geq \tilde{\alpha}$  and  $\mu$  is concave then  $\hat{\nu} = \hat{\alpha}\mu + (1 - \hat{\alpha})\tilde{\mu}$  is more ambiguity averse than  $\tilde{\nu} = \tilde{\alpha}\mu + (1 - \tilde{\alpha})\tilde{\mu}$ .*

**Proof.** The result follows from noting that since  $\mu$  is convex for all  $A \subseteq S$ ,  $\mu(A) \leq \tilde{\mu}(A)$ . ■

If  $\nu = \alpha\mu + (1 - \alpha)\tilde{\mu}$  is a JP capacity then

$$\int f d\nu = \alpha \min_{p \in \mathcal{C}(\mu)} \mathbf{E}_p f + (1 - \alpha) \max_{p \in \mathcal{C}(\mu)} \mathbf{E}_p f,$$

where  $\mathcal{C}(\mu)$  is the core of  $\mu$ . This equation provides an intuitive representation of ambiguity. When faced with ambiguity the decision-maker does not know the true probability distribution (if this concept is meaningful). Instead (s)he considers a number of probability distributions to be possible. If  $\alpha = 0$  the reaction is pessimistic since (s)he evaluates any given act by the least favourable probability distribution. Similarly if  $\alpha = 1$  the reaction to ambiguity is optimistic. In the general case, the decision-maker's reaction to ambiguity is in part pessimistic and in part optimistic.

A useful special case is the neo-additive capacity, defined below, which generates CEU preferences that display both optimism and pessimism. They are the simplest class of capacities with this property.

**Definition 2.13** *Let  $\alpha, \delta$  be real numbers such that  $0 < \delta < 1, 0 < \alpha < 1$ , define a neo-additive capacity  $\nu$  by  $\nu(A) = \delta(1 - \alpha) + (1 - \delta)\pi(A)$ ,  $\emptyset \subsetneq A \subsetneq S$ .<sup>8</sup>*

The neo-additive capacity describes a situation where the decision maker's 'beliefs' are represented by  $\pi$ . However (s)he has some doubts about these beliefs. The reaction to these doubts is in part pessimistic and in part optimistic. These preferences maintain a separation between ambiguity and ambiguity attitude which are measured by  $\delta$  and  $\alpha$  respectively. The highest possible level of ambiguity corresponds to  $\delta = 1$ , while  $\delta = 0$  corresponding to no ambiguity. Higher

---

and Kelsey and Nandeibam (1996). Although these papers differ in a number of respects, they agree that this is the most appropriate definition of when one capacity is more ambiguity-averse than another. The present paper focuses on the comparative statics of changes in ambiguity attitude. We are interested in when one set of preferences is more ambiguity-averse than another, not the absolute definition of ambiguity-aversion.

<sup>8</sup>Neo-additive is an abbreviation for Non-extremal outcome additive. Neo-additive capacities are axiomatised in Chateauneuf, Eichberger, and Grant (2005).

levels of  $\alpha$  corresponding to more ambiguity-aversion. Purely ambiguity-loving preferences are given by  $\alpha = 0$ , while the highest level of ambiguity-aversion occurs when  $\alpha = 1$ .

Chateauneuf, Eichberger, and Grant (2005), show that the Choquet expected value of a real valued function  $a : S_{-i} \rightarrow \mathbb{R}$  with respect to the neo-additive-capacity  $\nu$  is given by:

$$\int a d\nu = \delta \alpha \min_{s \in S_{-i}} a(s) + \delta (1 - \alpha) \max_{s \in S_{-i}} a(s) + (1 - \delta) \cdot \mathbf{E}_\pi a.$$

Thus the Choquet integral for a neo-additive capacity is a weighted averaged of the highest payoff, the lowest payoff and the expected payoff. The response to ambiguity partly optimistic represented by the weight given to the best outcome and partly pessimistic. In Chateauneuf, Eichberger, and Grant (2005) it is shown that CEU preferences with neo-additive capacities can also be represented in the following form:

$$\int a d\nu = (1 - \alpha) \max_{p \in \mathcal{P}} \mathbf{E}_p a + \alpha \min_{p \in \mathcal{P}} \mathbf{E}_p a,$$

where  $\nu = \delta (1 - \alpha) + (1 - \delta) \pi$  is a neo-additive capacity,  $\mathcal{P} = \{p \in \Delta(S_{-i}) : p \geq (1 - \delta) \pi\}$  and  $a \in A(S)$ . Thus  $\mathcal{P}$  is the set of measures ‘centred’ around a fixed  $\pi \in \Delta(S)$ .

Neo-additive capacities are restrictive since they only allow the best and worst outcome to be over-weighted. It is more plausible that ambiguity causes a number of good and bad outcomes to be over-weighted. It does not seem unreasonable to assume that in most cases the worst outcome is death. However people may also be concerned about other bad outcomes such as losing large amounts of money. Similarly optimism will intuitively result in a number of good outcomes being over-weighted.<sup>9</sup> Hence for most of this paper we consider more general JP-capacities, which represent ambiguity as over-weighting a number of good and bad outcomes.

### 3 EQUILIBRIUM

In this section we describe our solution concept and prove existence of equilibrium. First we discuss the support of ambiguous beliefs, which is a key concept for defining equilibrium in games. There have been a number of solution concepts for games with ambiguity, see for instance Dow and Werlang (1994), Lo (1996) or Marinacci (2000). In all of these, the support of

---

<sup>9</sup>One piece of evidence for this is that lotteries have more than one prize.

a player's beliefs is used to represent the set of strategies that (s)he believes his/her opponents will play. An equilibrium is defined to occur when every profile of strategies in the support consists only of best responses. The main difference between the various solution concepts is that they use different support notions. Thus the definition of support is important.

### 3.1 Support of Ambiguous Beliefs

In this section we propose a definition of the support of a capacity. It extends the previous literature since, as we argue in the appendix, many previous support notions implicitly assumed ambiguity-aversion. For *convex* capacities most support notions coincide. (See Appendix A for further discussion of the relation between different support notions.) We restrict attention to preferences which can be represented in both the CEU and multiple priors forms. Below we define the support for a convex capacity.

**Definition 3.1** *If  $\mu$  is a convex capacity on  $S_{-i}$ , we define the support of  $\mu$ ,  $\text{supp } \mu$ , by*

$$\text{supp } \mu = \bigcap_{p \in \mathcal{C}(\mu)} \text{supp } p.^{10}$$

We define the support of a JP-capacity to be the support of the convex capacity on which it is based.

**Definition 3.2** *If  $\nu = \alpha\mu + (1 - \alpha)\tilde{\mu}$  is a JP-capacity on  $S_{-i}$ , we define the support of  $\nu$ ,  $\text{supp } \nu$ , by  $\text{supp } \nu = \text{supp } \mu$ .*

The support always exists but may possibly be empty. As we shall demonstrate existence of equilibrium with a non-empty support, this possibility does not create a problem for our analysis. We shall now proceed to relate the support to the decision weights in the Choquet integral.

**Definition 3.3** *If  $\nu$  is a capacity on  $S_{-i}$ , define*

$$\mathcal{B}(\nu) = \{s \in S_{-i} : \forall A \subsetneq S_{-i}, s \notin A; \nu(A \cup s) > \nu(A)\}.$$

---

<sup>10</sup>This definition is essentially the same as the inner support as defined in Ryan (1997).

The set  $\mathcal{B}(\nu)$  consists of those states which always get positive weight in the Choquet integral no matter which act is being evaluated. To see, this, recall that the Choquet expected utility of a given act,  $a$ , is a weighted sum of utilities. The weight assigned to strategy profile  $\tilde{s}$  is  $\nu(\{s : a(s) \succ a(\tilde{s})\} \cup \{\tilde{s}\}) - \nu(s : a(s) \succ a(\tilde{s}))$ . These weights depend on the way in which the act ranks the states. Since there are  $n!$  ways the states can be ranked, in general there are  $n!$  profiles of decision weights used in evaluating the Choquet integral with respect to a given capacity. If  $\forall A \subsetneq S, s \notin A, \nu(A \cup \tilde{s}) > \nu(A)$  then the decision-weight on states  $\tilde{s}$  is positive no matter how state  $\tilde{s}$  is ranked by act  $a$ . The state space can be partitioned into three sets, those states which are given positive weight by all of the decision weights, those states given positive weight by some sets of decision weights but not others and those states which are given weight zero by all decision weights. Sarin and Wakker (1998) argue that the decision-maker's beliefs may be deduced from these decision weights. With this interpretation,  $\mathcal{B}(\nu)$  is the set of states in which the decision-maker 'believes' in the strong sense that they always get positive weight in the Choquet integral. Proposition 3.1 shows that the support consists of states in which the decision-maker believes in this sense. Similarly the states which always get zero weight are those which the decision-maker believes to be impossible. The remaining states are those which the decision-maker believes to be ambiguous. The weight they get in the Choquet integral may or may not be positive depending on the context.<sup>11</sup>

**Proposition 3.1** *Let  $\nu$  be a JP-capacity then  $\text{supp } \nu \subseteq \mathcal{B}(\nu)$ .*

**Proof.** Since  $\nu$  is a JP-capacity we may write  $\nu = \alpha\mu + (1 - \alpha)\tilde{\mu}$  for some convex capacity  $\mu$ . Take  $\hat{s} \in \text{supp } \nu$  and  $A \subsetneq S, \hat{s} \notin A$ . Let  $\hat{p} = \text{argmin}_{p \in \mathcal{C}(\mu)} p(A \cup \hat{s})$ . Because  $\mu$  is convex  $\mu(A \cup \hat{s}) - \mu(A) = \hat{p}(A \cup \hat{s}) - \mu(A) = \hat{p}(A) - \mu(A) + \hat{p}(\hat{s}) > 0$ , since  $\hat{p}(A) \geq \mu(A)$  and  $\hat{p}(\hat{s}) > 0$  by Proposition A.1.

Let  $\tilde{p} = \text{argmin}_{p \in \mathcal{C}(\mu)} p(S \setminus A)$ . Then  $\tilde{\mu}(A \cup \hat{s}) - \tilde{\mu}(A) = \mu(S \setminus A) - \mu(S \setminus (A \cup \hat{s})) \geq \tilde{p}(S \setminus A) - \tilde{p}(S \setminus (A \cup \hat{s})) = 1 - \tilde{p}(A) - [1 - \tilde{p}(A \cup \hat{s})] = \tilde{p}(A) + \tilde{p}(\hat{s}) - \tilde{p}(A) > 0$ , since  $\tilde{p}(\hat{s}) > 0$ . Hence for  $A \subsetneq S, s \notin A, \nu(A \cup \tilde{s}) > \nu(A)$ . ■

The following example demonstrates that it is not possible to prove a converse to Proposition 3.1.

---

<sup>11</sup>In earlier drafts of this paper we have shown that similar results can be obtained if  $\mathcal{B}(\nu)$  is used as the support notion. This shows that our results are reasonably robust.

**Example 1** Assume there are three states of nature,  $S = \{s_1, s_2, s_3\}$ . Consider capacities  $\nu, \mu$  defined on  $S$  in the table below:

	$\emptyset$	$s_1$	$s_2$	$s_3$	$s_1 s_2$	$s_1 s_3$	$s_2 s_3$	$S$
$\mu$	0	$\beta$	0	0	$\frac{1}{2}$	$\beta$	0	1
$\tilde{\mu}$	0	1	$1 - \beta$	$\frac{1}{2}$	1	1	$1 - \beta$	1
$\nu$	0	$1 - \alpha(1 - \beta)$	$(1 - \alpha)(1 - \beta)$	$\frac{1}{2}(1 - \alpha)$	$1 - \frac{1}{2}\alpha$	$1 - \alpha(1 - \beta)$	$(1 - \alpha)(1 - \beta)$	1

where  $\nu = \alpha\mu + (1 - \alpha)\tilde{\mu}$  and  $0 < \alpha < 1$ ,  $0 < \beta < \frac{1}{2}\alpha$ . Then  $\text{supp } \nu = \{s_1\}$  and  $\mathcal{B}(\nu) = \{s_1, s_2\}$ ,  $\text{supp } \mu = \{s_1\}$ ,  $\mathcal{B}(\mu) = \{s_1\}$ .

**Proof.** Since  $\mu$  is convex, it follows from Proposition A.1, that  $\text{supp } \mu = \{s_1\}$ . By definition  $\text{supp } \nu = \text{supp } \mu$ .

By Proposition 3.1,  $s_1 \in \mathcal{B}(\nu)$ . We can show  $s_2 \in \mathcal{B}(\nu)$ , since  $\nu(\emptyset \cup s_2) = 1 - \alpha > \nu(\emptyset)$ ,  $\nu(s_1 \cup s_2) = 1 - \frac{1}{2}\alpha > \nu(s_1) = 1 + \alpha\beta - \alpha$ ,  $\nu(s_3 \cup s_2) = (1 - \beta)(1 - \alpha) > \nu(s_3) = \frac{1}{2}(1 - \alpha)$ ,  $\nu(s_1 s_3 \cup s_2) = 1 > \nu(s_1 s_3) = 1 + \alpha\beta - \alpha$ . However  $s_3 \notin \mathcal{B}(\nu)$ , since  $\nu(s_1 s_3) = \nu(s_1)$ . ■

**Jürgen you claimed there were some mistakes in this example. Is it OK now?**

Since  $\mu$  is a belief function,  $\nu$  is a JP capacity. Consider the capacity  $\nu_\alpha = \alpha\mu + (1 - \alpha)\tilde{\mu}$ , where  $\mu$  is a convex capacity. Intuitively as  $\alpha$  changes, ambiguity attitude changes, while beliefs and perceived ambiguity remain constant. If the  $\mathcal{B}(\nu)$  is taken to represent beliefs, one has to argue that the decision-maker's beliefs change as his/her ambiguity attitude changes. However  $\text{supp } \nu_\alpha$  is independent of  $\alpha$ . This suggests to us that  $\text{supp } \nu$  is a superior notion of support to  $\mathcal{B}(\nu)$ , since it makes a clear distinction between ambiguity-attitude and beliefs. In addition the fact that  $\text{supp } \nu$  is the intersection of support of the probabilities in the core is quite intuitive.

### 3.2 Definition of Equilibrium

We define an equilibrium to be a situation where each player maximises his/her (Choquet) expected utility given his/her beliefs. In addition beliefs have to be reasonable in the sense that each player believes that his/her opponents play best responses. We interpret this requirement as implying the support of any given player's beliefs should consist of best responses for the other players. Denote by  $R_i(\nu_i) = \arg \max \{P_i(s_i, \nu_i) \mid s_i \in S_i\}$  the best response correspondence of player  $i$  given beliefs  $\nu_i$ .

**Definition 3.4** An  $n$ -tuple of capacities  $\nu^* = \langle \nu_1^*, \dots, \nu_n^* \rangle$  is an equilibrium under ambiguity if:

$$\forall i, \emptyset \neq \text{supp } \nu_i^* \subseteq \times_{j \neq i} R_j(\nu_j^*).$$

If  $\forall i, k_i^* \in \text{supp } \nu_i^*$ , we say that  $k^* = \langle k_1^*, \dots, k_n^* \rangle$  is an equilibrium strategy profile.

In equilibrium, the beliefs of player  $i$  are represented by a capacity  $\nu_i^*$ , whose support consists of strategies that are best responses for his/her opponents. A player's evaluation of a particular strategy may, in part, depend on strategies of his/her opponents which do not lie in the support. We interpret these as events a player views as unlikely but which cannot be ruled out. This may reflect some doubts (s)he may have about the rationality of the opponents or whether (s)he correctly understands the structure of the game.

**Definition 3.5** Let  $\Gamma$  be a game and let  $\hat{\nu} = \langle \hat{\nu}_1, \dots, \hat{\nu}_n \rangle$  be an equilibrium of  $\Gamma$ , if  $\forall i, \text{supp } \hat{\nu}_i$  contains a single strategy profile we say that it is pure, otherwise we say that it is mixed.

Players choose pure strategies and do not randomise. Hence we are not able to interpret a mixed equilibrium as a randomisation. In a mixed equilibrium some player  $i$  say, will have two or more best responses. The support of other players' beliefs about  $i$ 's play, will contain some or all of them. Thus an equilibrium, where the support contains multiple strategy profiles, is an equilibrium in beliefs rather than randomisations. If, in addition, it is required that beliefs are additive and then a pure equilibrium is a Nash equilibrium.

### 3.3 Existence of Equilibrium

We shall confine attention to situations where beliefs may be represented by JP-capacities. The reason for this is that this is a large class of capacities in which it is possible to specify ambiguity and the ambiguity-attitude exogenously. This enables us to study the comparative statics of changing ambiguity-attitude in games. The class of capacities, defined below, are a set of candidate equilibria for games, which allow us to vary ambiguity and ambiguity-attitude exogenously.

**Definition 3.6 (Constant Contamination CC)** A capacity on  $S_{-i}$  is said to display constant contamination if it may be written in the form  $\nu_i = \nu_i(\pi_i, \chi_i, \delta_i, \alpha_i) = \delta_i \pi_i(A) +$

$(1 - \delta_i) [\alpha_i \chi_i(A) + (1 - \alpha_i) \tilde{\chi}_i(A)]$ , where  $\pi_i$  is an additive probability distribution and  $\chi_i$  is a convex capacity with  $\text{supp } \chi_i = \emptyset$ . If  $\pi_i^{\tilde{s}}$  denotes the probability distribution on  $S_{-i}$ , which assigns probability 1 to  $\tilde{s}_{-i}$  we shall write  $\nu_i^{\tilde{s}} = \nu_i(\pi_i^{\tilde{s}}, \chi_i, \delta_i, \alpha_i)$ . We shall suppress the arguments  $(\alpha_i, \delta_i, \chi_i)$  when it is convenient.

Thus  $\nu_i(\pi_i, \chi_i, \delta_i, \alpha_i)$  describes a situation where player  $i$  ‘believes’ that his/her opponents will play the mixed or pure strategy profile described by  $\pi_i$  but lacks confidence in this belief. The CC-capacity has a separation between beliefs represented by  $\pi$ , ambiguity represented by  $\chi$  and  $\delta$  and ambiguity attitude represented by  $\alpha$ . The situation where  $i$  believes that his/her opponents will play the pure strategy profile  $\tilde{s}$  is described by the capacity  $\nu_i^{\tilde{s}}$ . The parameter  $\delta$  determines the weight the individual gives to ambiguity. Lower values of  $\delta$  correspond to more ambiguity hence  $\delta$  may be interpreted as a measure of confidence in the decision-maker’s probabilistic belief  $\pi$ . The capacity  $\chi$  determines which strategy profiles the player regards as ambiguous. CC-capacities are a special case of JP-capacities. The following result finds the support of a CC capacity.

**Proposition 3.2** *Let  $\nu = \delta\pi(A) + (1 - \delta)[\alpha\chi(A) + (1 - \alpha)\hat{\chi}(A)]$  be a CC capacity. Then  $\text{supp } \nu = \text{supp } \pi$ .*

**Proof.** Let  $\mu = \delta\pi(A) + (1 - \delta)\chi(A)$ . Then by Lemma B.2,  $\nu = \alpha\mu + (1 - \alpha)\hat{\mu}$  is a JP capacity. By definition  $\text{supp } \nu = \text{supp } \mu$ . By Lemma B.2,  $\text{supp } \mu = \text{supp } \pi$ . ■

**Remark 1** *If  $\nu_0$  denotes the complete uncertainty capacity, then a neo-additive capacity may be written in the form  $\nu = \delta\pi(A) + (1 - \delta)[\alpha\nu_0(A) + (1 - \alpha)\tilde{\nu}_0(A)]$ . Proposition B.2 implies that  $\text{supp } \nu = \text{supp } \pi$ . Recall we interpret a neo-additive capacity as describing a situation the decision-maker’s beliefs are represented by the additive probability distribution  $\pi$ , however (s)he may lack confidence in this belief. Given this, it seems intuitive that the support of the neo-additive capacity should coincide with the support of  $\pi$ . This provides an argument in favour of our definition of support, since it gives the intuitively correct result for neo-additive capacities.*

In game theory it is common to assume that each player believes that his/her opponents act independently. This can be achieved if we assume that the capacity  $\mu$  is an independent product.<sup>12</sup>

---

<sup>12</sup>Technically we need to assume that  $\mu$  is a Möbius independent product of belief functions defined on the marginals. For a definition of the Möbius independent product and further discussion see Ghirardato (1997).



We commence the analysis of equilibria with ambiguity by defining the marginal benefit under ambiguity.

**Definition 3.7 *Marginal Benefit*** Suppose that player  $i$  has beliefs described by a capacity  $\nu_i$  on  $S_{-i}$ . Define

$$\text{MB}_i(s_i, \nu) = \int u_i(s_i, s_{-i}) d\nu_i(s_{-i}) - \int u_i(s_i - 1, s_{-i}) d\nu_i(s_{-i}).$$

We interpret  $\text{MB}_i(s_i, \nu)$  as player  $i$ 's perceived marginal benefit from increasing his/her strategy from  $s_i - 1$  to  $s_i$ , given that (s)he has beliefs represented by the capacity  $\nu$ . In general this will be different to the marginal benefit without ambiguity from Definition 2.3.

The following result demonstrates existence of equilibrium. We take the ambiguity represented by  $\chi_i$  and  $\delta_i$ , and ambiguity attitude represented by  $\alpha_i$  as given. In games of aggregate externalities with strategic complementarity we are able to demonstrate the existence of pure equilibria.

**Theorem 3.1** Let  $\Gamma$  be a game of positive aggregate externalities with strategic complements. Then for any given  $n$ -tuples  $\alpha = \langle \alpha_1, \dots, \alpha_n \rangle$ ,  $\delta = \langle \delta_1, \dots, \delta_n \rangle$  and  $\chi = \langle \chi_1, \dots, \chi_n \rangle$ ,  $\Gamma$  has a pure equilibrium in CC capacities.

### 3.4 Symmetric Games

In symmetric games we can strengthen the previous result by proving existence of symmetric equilibrium. For this, we require that the strategy sets and payoff functions be symmetric. Moreover we need to ensure that players perceive ambiguity in a symmetric way. This is captured by the following definition.

**Notation 3.1** Let  $\psi : \{1, \dots, n\} \setminus i \rightarrow \{1, \dots, n\} \setminus i$  be a permutation. Then  $\psi$  induces a map  $\tilde{\psi} : S_{-i} \rightarrow S_{-i}$  by  $\tilde{\psi}(\langle s_1, \dots, s_n \rangle) = \langle s_{\psi(1)}, \dots, s_{\psi(n)} \rangle$ .

**Definition 3.8** A capacity  $\zeta$  on  $S_{-i}$  is said to be anonymous if for any permutation  $\psi : \{1, \dots, n\} \setminus i \rightarrow \{1, \dots, n\} \setminus i$  and all  $A \subseteq S_{-i}$ ,  $\zeta(\tilde{\psi}(A)) = \zeta(A)$ .<sup>13</sup>

---

<sup>13</sup>We refer to such capacities as *anonymous* rather than *symmetric*, since the term symmetric is commonly used to denote capacities which satisfy  $\nu(A) + \nu(\neg A) = 1$ , see Gilboa (1989).

**Definition 3.9** Define  $\xi : \{1, \dots, n\} \setminus i \rightarrow \{1, \dots, n\} \setminus j$  by  $\xi(j) = i$  and  $\xi(k) = k, k \neq j$ . Capacities  $\chi_i$  on  $S_{-i}$  and on  $S_{-j}$  are said to be similar if for all  $A \subseteq S_{-i}, \chi_j(\xi(A)) = \chi_i(A)$ .

**Definition 3.10** Let  $\Gamma$  be a game and let  $\nu^* = \langle \nu_1^*, \dots, \nu_n^* \rangle$  be an equilibrium of  $\Gamma$ . The profile of capacities,  $\nu^*$  is said to be a symmetric equilibrium, if for all  $i, j, \nu_i^*$  is similar to  $\nu_j^*$ .

In a symmetric equilibrium, all players have the same best responses and have correspondingly similar beliefs about what their opponents will do. The next result demonstrates the existence of symmetric equilibria in symmetric games. Compared to Theorem 3.1 it is stronger in that it does not assume strategic complementarity. However the cost of relaxing this assumption is that we are only able to demonstrate the existence of mixed equilibria.

**Theorem 3.2 (Existence of symmetric equilibrium)** Let  $\Gamma$  be a symmetric game, then for any given numbers  $\alpha, \delta, 0 \leq \alpha, \delta \leq 1$  and any anonymous capacity  $\chi$  with  $\text{supp } \chi = 0$ ,  $\Gamma$  has a symmetric equilibrium  $\nu_i = \nu(\pi, \chi, \delta, \alpha)$  in CC capacities.

## 4 COMPARATIVE STATICS

In this section we investigate the comparative statics of changes in ambiguity-attitude on equilibrium.

### 4.1 Strategic Complements

For games of positive aggregate externalities with strategic complements, an increase in ambiguity-aversion increases equilibrium strategies. Intuitively if a given player becomes more ambiguity-averse (s)he will place more weight on outcomes which are perceived as bad. If there are positive externalities, a bad outcome would be when an opponent plays a low strategy. In the presence of strategic complementarity if a given player increases the decision weight on low strategies of his/her opponents this will reduce his/her incentive to play a high strategy.

The following theorem is our comparative result on games with strategic complementarity. It shows that an increase in pessimism will reduce the equilibrium strategies in games with positive aggregate externalities and strategic complements. If there are multiple equilibria, the strategies played in the highest and lowest equilibria will decrease. For this result we assume

that the ambiguity-attitude of one player changes, the ambiguity-attitudes of other players and the perceived ambiguity measured by  $\delta$  and  $\chi$  are held constant.

**Theorem 4.1** *Let  $\Gamma$  be a game of positive aggregate externalities with strategic complements. Assume that players' beliefs may be represented by CC capacities  $\nu_i(\pi_i, \chi_i, \delta_i, \alpha_i)$ . Let  $\underline{k}'$  (resp.  $\bar{k}'$ ) denote the lowest (resp. highest) equilibrium strategy profile when the degree of ambiguity-aversion is  $\alpha^* = \langle \alpha_1^*, \dots, \alpha_n^* \rangle$ . Let  $\underline{k}^*$  (resp.  $\bar{k}^*$ ) denote the lowest (resp. highest) equilibrium strategy profile when the degree of ambiguity-aversion is  $\alpha' = \langle \alpha_1', \dots, \alpha_n' \rangle$ . If  $\alpha^* > \alpha'$ , then  $\underline{k}' \geq \underline{k}^*$  and  $\bar{k}' \geq \bar{k}^*$ .*

**Proof.** Assume first that  $\alpha^*$  and  $\alpha'$  only differ in one component, i.e.  $\exists i, \forall j \neq i, \alpha_j^* = \alpha_j'$ . By Lemma B.4,  $\forall i, \text{MB}_i(s_i, \nu_i^{\tilde{s}}(\alpha_i', \delta_i, \chi_i)) < 0$  for  $\bar{k}' < \tilde{s} \leq m^*$ . By Lemma B.5,  $\forall i, \text{MB}_i(s_i, \nu_i^{\tilde{s}}(\alpha_i', \delta_i, \chi_i)) \geq \text{MB}_i(s_i, \nu_i^{\tilde{s}}(\alpha_i^*, \delta_i, \chi_i))$ . Hence  $\forall i, \text{MB}_i(s_i, \nu_i^{\tilde{s}}(\alpha_i^*, \delta_i, \chi_i)) < 0$ , for  $\bar{k}' < \tilde{s} \leq m^*$ . However,  $\text{MB}_i(s_i, \nu_i^{\tilde{s}}(\alpha_i^*, \delta_i, \chi_i)) \geq 0$  is a necessary condition for there to be an equilibrium in which  $s$  is the equilibrium strategy. This implies  $\bar{k}^* \leq \bar{k}'$ . A similar argument applies to the smallest equilibrium.

The general result follows by repeated application of the result where  $\alpha^*$  and  $\alpha'$  only differ in one component. ■

**Remark 2** *The comparative statics are reversed in games of negative aggregate externalities, for further details see Eichberger and Kelsey (2002).*

## 4.2 Strategic Complementarity and Multiple Equilibria

Strategic Complementarity can give rise to multiple Nash equilibria. Under some assumptions we can show if there are multiple equilibria without ambiguity, if there is sufficient optimism (resp. pessimism) equilibrium will be unique and will correspond to the highest (resp. lowest) equilibrium without ambiguity. Thus ambiguity has a double effect. Below we propose a measure of how ambiguous a capacity is.

**Definition 4.1** *The maximal degree of ambiguity of capacity  $\mu$  is defined by:*

$$\lambda(\mu) = 1 - \min_{A \subseteq S} (\mu(A) + \mu(\neg A)).$$

The next result shows that if players are sufficiently optimistic, equilibrium is unique and is higher than the highest equilibrium without ambiguity. To prove this we need the following assumption.

**Assumption 4.1** *For  $1 \leq i \leq n$ ,  $u_i(s_i, \bar{s}_{-i})$  has a unique maximiser, i.e.*

$$|\operatorname{argmax}_{s_i \in S_i} u_i(s_i, \bar{s}_{-i})| = 1 .$$

This assumption is required for technical reasons. If the strategy space were continuous it would be implied by our other assumptions. It says that the gaps in the discrete strategy space do not fall in the wrong place. The following result is a corollary of Theorem 4.1 and Lemma B.7.

**Proposition 4.1** *Consider a game of positive aggregate externalities with strategic complements which satisfies Assumption 4.1. There exist  $\bar{\alpha}$  and  $\bar{\lambda}$  such that if the minimal degree of ambiguity is  $\lambda(\mu_i) \geq \bar{\lambda}$  and  $\alpha_i \geq \bar{\alpha}$ , (resp.  $\leq \underline{\alpha}$ ) for  $1 \leq i \leq n$ , equilibrium is unique and is larger (resp. smaller) than the largest (resp. smallest) equilibrium without ambiguity.*

In a game with strategic complements with multiple Nash equilibria, increasing ambiguity causes the multiplicity of equilibria to disappear while increasing ambiguity-aversion causes the equilibrium strategies to decrease. Thus ambiguity and ambiguity-attitude have distinct effects. Combined with sufficient optimism, ambiguity can cause the economy to move to a higher level equilibrium.

### 4.3 Strategic Substitutes

Next we study games with strategic substitutes. For this section, we shall assume that all games are symmetric, since there are no general comparative statics results for strategic substitutes. However it is possible to obtain a result for symmetric equilibria of symmetric games. The next result shows that in games with strategic substitutes and positive externalities, increasing ambiguity-aversion raises the strategy played in symmetric equilibrium.

**Proposition 4.2** *Consider a game of positive aggregate externalities with strategic substitutes. Let  $\bar{\nu} = \nu(\pi, \mu, \delta, \bar{\alpha})$  (resp.  $\hat{\nu} = \nu(\pi, \mu, \delta, \hat{\alpha})$ ) be a symmetric equilibrium in CC-capacities*

in which  $\bar{k}$  (resp.  $\hat{k}$ ) is the highest strategy played and  $\bar{\ell}$  (resp.  $\hat{\ell}$ ) is the lowest strategy played. Then  $\hat{\alpha} > \bar{\alpha}$  implies  $\hat{k} \geq \bar{k}$  and  $\hat{\ell} \geq \bar{\ell}$ .<sup>14</sup>

With positive externalities, an increase in ambiguity-aversion increases the weight on lower strategies of the opponent. In a game of strategic substitutes this will increase the perceived marginal benefit of a given player's own action and hence (s)he will have a higher best response to any given strategy profile of his/her opponents. As a result the strategy played in a symmetric equilibrium will increase.

## 5 OVERCONFIDENCE

### 5.1 Comparative Statics of Overconfidence

In this section we study the comparative statics of overconfidence in games. The evidence suggests that overconfidence is common and is more prevalent when the outcome depends on an individual's own performance as well as random factors. We consider stochastic games where pay-offs depend on a state of nature as well as the strategies chosen. In this case there is an objective probability available. However players have subjective probabilities which differ from the objective probability by giving higher probabilities to favourable outcomes. We believe that stochastic games are a good way to model overconfidence, since players' rewards depend on their own strategic choice and on luck.

Consider a stochastic game  $\Gamma = \langle I, \Theta, (S_i) (u^i) : 1 \leq i \leq n \rangle$ , where player  $i$ 's strategy set  $S_i \subseteq \mathbb{R}$  is a compact interval. As usual we denote the set of possible strategy profiles for players other than  $i$  by  $S_{-i}$ . We define an order on  $S$  by  $\hat{s} \geq \tilde{s} \Leftrightarrow \hat{s}_i \geq \tilde{s}_i$  for  $1 \leq i \leq n$  and  $\hat{s} > \tilde{s}$  if  $\hat{s} \geq \tilde{s}$  and  $\exists i, \hat{s}_i > \tilde{s}_i$ . Similarly we may define a partial ordering on  $S_{-i}$ , which by a slight abuse of notation, we also denote by  $\geq$ . We use lattice-theoretic techniques to analyse the comparative statics of this model.<sup>15</sup> The pay-off function  $u^i(s_i, s_{-i}, \theta)$  of player  $i$  depends on his/her own strategy, his/her opponents' strategies and on the state of nature  $\theta$ .

**Definition 5.1** *Let  $X$  be a lattice and  $T$  a partially ordered set. The function  $\psi(x, t) : X \times T \rightarrow \mathbb{R}$ , has (strictly) increasing differences if whenever  $\hat{t} > \tilde{t}$ ,  $\psi(x, \hat{t}) - \psi(x, \tilde{t})$  is a (strictly)*

<sup>14</sup>Since we are not able to prove existence of a pure equilibrium, the equilibrium strategies are not necessarily unique hence we need to consider the highest and lowest equilibrium strategies.

<sup>15</sup>For further details see, Milgrom and Roberts (1990), Milgrom and Roberts (1994), Milgrom and Shannon (1994), Topkis (1998) and Vives (1999).

increasing function of  $x$ .

We assume that for any given  $\theta$ ,  $\Gamma$  is a supermodular game.

**Assumption 5.1** *For all  $i : 1 \leq i \leq n$ ,  $u^i(s_i, s_{-i}, \theta)$  is supermodular in  $s_i$  and displays strictly increasing differences in  $(s_i, s_{-i})$ .*

This says that if player  $i$ 's opponents play a higher strategy, the (s)he has a greater incentive to increase his/her own strategy. The state of nature,  $\theta$ , is drawn from a finite set  $\Theta = \{\theta_1, \dots, \theta_J\}$ . Player  $i$  has beliefs over  $\Theta$  described by the probability distribution  $\pi^i \in \Pi^i$ , where  $\Pi^i$  is a set of probability distributions over  $\Theta$ , which is partially ordered by first order stochastic dominance. Let  $\Pi = \times_{i=1}^n \Pi^i$ . If  $\hat{\pi}, \tilde{\pi} \in \Pi$ , we say  $\hat{\pi} \geq \tilde{\pi}$  if  $\forall i, \hat{\pi}_i \geq \tilde{\pi}_i$ .

**Assumption 5.2** *The utility of player  $i$ ,  $u^i(s_i, s_{-i}, \theta)$ , is strictly increasing in  $\theta$  and has strictly increasing differences in  $s_i$  and  $\theta$ .*

A higher  $\theta$  increases a player's utility and increases his/her incentive to play a higher strategy. If  $u^i$  were differentiable this would imply that total and marginal utility is increasing in  $\theta$ .

**Definition 5.2** *Let  $\bar{\rho}^i(s_{-i}, \pi^i) = \max \operatorname{argmax}_{s_i \in S_i} \sum_{j=k}^J \pi_j^i u_1^i(s_i, s_{-i}, \theta_j)$  (resp.  $\underline{\rho}^i(s_{-i}, \pi^i) = \min \operatorname{argmax}_{s_i \in S_i} \sum_{j=k}^J \pi_j^i u_1^i(s_i, s_{-i}, \theta_j)$ ) denote the largest (resp. smallest) best response of player  $i$ . Let  $\bar{\rho}(s, \pi) = \langle \bar{\rho}^1(s_{-1}, \pi^1), \dots, \bar{\rho}^n(s_{-n}, \pi^n) \rangle$ .*

**Theorem 5.1** *The set of pure strategy Nash equilibria is non-empty and has largest and smallest elements.*

**Proof.** Theorem 2.3 of Vives (1999) implies that the largest and smallest best response functions of player  $i$  are increasing functions of  $s_{-i}$  and  $\pi^i$ . The existence of largest and smallest equilibria follows by applying Tarski's fixed point theorem, Tarski (1955), to the largest and smallest best response correspondences. ■

Say that player  $i$  has an *increase in confidence* if he has a new subjective probability  $\tilde{\pi}^i$  which first order stochastically dominates the previous one,  $\pi^i$ . The following result says that an increase in confidence increases equilibrium strategies.

**Theorem 5.2** *Let  $\Gamma$  be a game with strategic complements. Consider a given player,  $i$ , say. An increase in confidence for player  $i$  increases the strategies played in the largest and smallest equilibrium.*

**Proof.** By Lemma B.8, the largest best response  $\bar{\rho}^i(s_{-i}, \pi^i)$  of player  $i$  is increasing in  $\pi^i$ , hence  $\bar{\rho}(s, \pi^i)$  is also increasing in  $\pi^i$ . The largest equilibrium strategy profile  $\bar{s}(\pi)$  satisfies  $\bar{s}(\pi) = \sup \{s \in S : \bar{\rho}(s, \pi^i) \geq s\}$ . Hence  $\bar{s}(\pi)$  is increasing in  $\pi$ . The proof for the smallest equilibrium is similar. ■

Thus if there is an objective probability distribution  $p$  on  $\Theta$  and individuals are overconfident then equilibrium strategies will be higher than otherwise. Overconfidence causes individuals to increase the weight they place on high values of  $\theta$  in their decision making. This increases their incentive to play a high strategy. Strategic complementarity implies that when a player increases his/her strategy his/her rivals have an incentive to play a higher strategy as well. Thus in equilibrium all play higher strategies.

## 5.2 The Rank Dependent Model

In this section we show that similar results may be obtained if players have RDEU preferences, see Quiggin (1982) and Yaari (1987). There is psychological evidence that individuals subjectively distort probabilities, see Gonzalez and Wu (1999). RDEU has been proposed to model this.

Let  $\Theta$  denote the set of states of nature and let  $p$  be an objective probability distribution on  $\Theta$ . For simplicity we shall assume that the outcomes are amounts of money, however the results can be easily extended to more general outcome spaces. RDEU assumes that the decision maker has a utility function  $u : \mathbb{R} \rightarrow \mathbb{R}$  and (s)he maximises a Choquet integral of utility with respect to a subjective distortion of the true probability  $\phi(p)$ . The distortion function  $\phi : [0, 1] \rightarrow [0, 1]$  is assumed to be strictly increasing and to satisfy  $\phi(0) = 0$  and  $\phi(1) = 1$ . Under these assumptions  $\phi(p)$  is a capacity on  $\Theta$ . Let  $a : \Theta \rightarrow \mathbb{R}$  be an act such that  $u(a(\theta_J)) \geq u(a(\theta_{J-1})) \geq \dots \geq u(a(\theta_1))$ . (Note this can always be ensured by relabelling the states.) RDEU postulates that individuals have preferences of the following form:

$$V(a) = u(a(\theta_J))\phi(p_J) + \sum_{j=1}^{J-1} u(a(\theta_j))[\phi(p_j) - \phi(p_{j+1})],$$

where  $p_j = p\{\theta_j, \dots, \theta_J\}$ .<sup>16</sup>

---

<sup>16</sup>This is slightly different to the original version of RDEU in Quiggin (1982) since we represent preferences as a Choquet integral, while Quiggin represents them as a dual Choquet integral. The difference is unimportant and is largely a matter of notation.

Within the RDEU model, there are two possible sources of risk aversion. If the utility function  $u$  is concave this will induce risk averse behaviour for the usual reasons. However the probability distortion function  $\phi$  may also cause risk averse behaviour. This is referred to as *probabilistic risk aversion* to differentiate it from conventional risk aversion which arises from diminishing marginal utility. In Wakker (1994) it is shown that if  $\hat{\succsim}$  and  $\tilde{\succsim}$  are two sets of RDEU preferences with the same utility function and distortion functions  $\hat{\phi}$  and  $\tilde{\phi}$  respectively, then  $\tilde{\succsim}$  is more risk-averse than  $\hat{\succsim}$  if and only if  $\tilde{\phi} = \psi \circ \hat{\phi}$  for some convex function  $\psi$ . In the present paper we consider behaviour in a game where the utility function is given. Thus the only source of changes in risk-aversion is from changes in  $\phi$ .

As before consider a stochastic game  $\Gamma = \langle I, \Theta, (S_i) (u^i) : 1 \leq i \leq n \rangle$ . Now assume that there is a known objective probability distribution  $p$  over  $\Theta$ . If the players satisfy the RDEU axioms, then player  $i$  would have preferences given by:

$$U^i(s_i, s_{-i}) = u^i(s_i, s_{-i}, \theta_J) \phi_i(p_J) + \sum_{j=1}^{J-1} u^i(s_i, s_{-i}, \theta_j) [\phi_i(p_j) - \phi_i(p_{j+1})],$$

where  $p_j = p\{\theta_j, \dots, \theta_J\}$  and  $\phi_i$  denotes the distortion function of individual  $i$ .

The following result shows how a change in the distortion function affects the equilibrium strategies in a game.

**Theorem 5.3** *Let  $\Gamma = \langle I, \Theta, (S_i) (u^i) : 1 \leq i \leq n \rangle$  be a stochastic game, where players have RDEU preferences and let  $\hat{\phi}_i$  and  $\tilde{\phi}_i$  be two distortion functions for individual  $i$  such that for all  $x \in (0, 1)$   $\hat{\phi}_i(x) > \tilde{\phi}_i(x)$ . The strategies played in the largest (resp. smallest) equilibrium with  $\hat{\phi}_i$  are greater than the strategies played in the largest (resp. smallest) equilibrium with  $\tilde{\phi}_i$ .*

**Proof.** Define two probability distributions  $\hat{\pi}$  and  $\tilde{\pi}$  on  $\Theta$  by  $\hat{\pi}(\theta_J) = \hat{\phi}_i(p_J)$ ,  $\hat{\pi}(\theta_j) = \hat{\phi}_i(p_j) - \hat{\phi}_i(p_{j+1})$ ,  $1 \leq j \leq J-1$  and  $\tilde{\pi}(\theta_J) = \tilde{\phi}_i(p_J)$ ,  $\tilde{\pi}(\theta_j) = \tilde{\phi}_i(p_j) - \tilde{\phi}_i(p_{j+1})$ ,  $1 \leq j \leq J-1$ . If the decision-maker has probability distortion function  $\hat{\phi}_i$  (resp.  $\tilde{\phi}_i$ ) then  $\hat{\pi}$  and (resp.  $\tilde{\pi}$ ) are the weights in the RDEU functional. We claim that  $\hat{\pi}$  first order stochastically dominates  $\tilde{\pi}$ . If  $1 \leq k \leq J-1$ ,  $\sum_{j=k}^J \hat{\pi}(\theta_j) = \hat{\pi}(\theta_J) + \sum_{j=k}^{J-1} \hat{\phi}_i(p_j) - \hat{\phi}_i(p_{j+1}) = \hat{\phi}_i(p_k) > \tilde{\phi}_i(p_k) = \sum_{j=k}^J \tilde{\pi}(\theta_j)$  and  $\hat{\pi}(\theta_J) = \hat{\phi}_i(p_J) > \tilde{\phi}_i(p_J) = \tilde{\pi}(\theta_J)$ , which establishes first order stochastic dominance. The result then follows from Theorem 5.2. ■

The following corollary says that if one or more individuals experience a increase in proba-



bilistic risk-aversion then equilibrium strategies will decrease.

**Corollary 5.1** *Let  $\Gamma = \langle I, \Theta, (S_i) \rangle$  ( $u^i$ ) :  $1 \leq i \leq n$  be a stochastic game and let  $\hat{\phi}_i$  and  $\tilde{\phi}_i$  be two RDEU distortion functions for individual  $i$  such that  $\tilde{\phi}_i = \psi \circ \hat{\phi}_i$  for some convex function  $\psi$ . The strategies played in the largest and smallest equilibrium with  $\hat{\phi}_i$  are greater than the strategies played in the largest and smallest equilibrium with  $\tilde{\phi}_i$ .*

**Proof.** In this case  $\tilde{\phi}(x) = \psi \circ \hat{\phi}(x) = \psi(\hat{\phi}(x) \cdot 1 + (1 - \hat{\phi}(x)) \cdot 0) \geq \hat{\phi}(x) \psi(1) + 1 - \hat{\phi}(x) \psi(0) = \hat{\phi}(x)$ . Thus Theorem 5.3 implies that a decrease in risk aversion by any player or group of players will result in an increase in equilibrium strategies. ■

This result is about comparative risk-aversion hence it does not require individuals to be risk-averse in any absolute sense. It would equally apply to somebody who was initially risk-seeking but became less risk-seeking. Although we have assumed that all individuals have the same objective probability distribution over  $\Theta$ , similar analysis would apply if individuals had heterogeneous subjective priors over  $\Theta$  and subjectively distorted these priors.

Suppose that a given individual,  $i$  say, becomes more risk averse. Then (s)he will place more weight on lower values of  $\theta$ . At low values of  $\theta$  the marginal benefit of playing a higher strategy is lower. Hence  $i$  will play a lower strategy. Since this is a game of strategic complements, other individuals will respond by playing lower strategies which induces  $i$  to reduce his/her strategy further.

### 5.3 Related Literature

There is a small extant literature on the economic implications of overconfidence. In Gervais, Heaton, and Odean (2000) the implications of overconfidence for investment decisions are studied. Managers and shareholders interests are not perfectly aligned, since shareholders can diversify their risk while managers cannot. Thus other things equal, shareholders would like to under take more risky projects than managers. In this situation it may be advantageous to hire an overconfident manager, since such a manager will have interests more closely aligned with those of shareholders.

In Van Den Steen (2002) it is argued that overconfidence can be produced by an effect similar to the winner's curse. Imagine a group of individuals with heterogeneous priors, each of whom has to make an independent choice from a set of available projects. If each chooses

the project which according to his/her prior has the greatest chance of success, then (s)he will be choosing the project for which his/her prior is most biased upwards. The result is that the equilibrium exhibits overconfidence, in the sense that each individual's belief his/her own project will succeed, is greater than the median individual's belief that the same project will succeed. This offers an explanation of why there is apparent overconfidence, since in this model each individual is fully rational. Nevertheless their actions appear overconfident from the point of view of others.

In Gervais and Goldstein (2004) it is shown that if one member of a team is over-confident in the sense of over-estimating his/her abilities the problem of moral hazard in teams is reduced. This can make all members of the team better-off including the overconfident individual. Like our result this arises because strategic complementarity between team members causes other individuals to raise their output.

## 6 CONCLUSION

### 6.1 Summary

This paper has examined two ways of modelling optimism or overconfidence in games. Many results are similar, an increase in either overconfidence or optimism will increase equilibrium strategies in games of strategic complements. The common theme is that in all cases possible courses of action are evaluated by a weighted sum of utilities. The changes in preference caused these weights to shift in a systematic way thus giving rise to a predictable comparative static effect.

Although similar, the two models use different assumptions and neither can be compared with the other. The comparative statics of stochastic games with either overconfidence or RDEU preferences can be analysed with assumptions which, are in a number of respects, weaker than those needed for games with ambiguity. We do not need to assume concavity in own strategy nor do we need to restrict attention to games with aggregate externalities. For games with ambiguity it was necessary to define a new equilibrium concept. In contrast, overconfidence can be modelled as a standard Nash equilibrium. An equilibrium with overconfidence differs from conventional game theory since players have subjective beliefs which disagree with the objective probabilities. However this difference is a change in the preferences not a change in

the solution concept. On the other hand it was not necessary to assume the existence of any extrinsic uncertainty in games with ambiguity. All uncertainty concerned the behaviour of other players.

One of the most important differences between the different theories is that while ambiguity preference and overconfidence can both cause equilibrium strategies to be higher than in Nash equilibrium, only ambiguity can have the effect of moving the system to a higher level equilibrium when there are multiple equilibria. In general overconfidence does not have a similar implication. If the game in the highest state of nature,  $\Gamma(\theta_J)$ , has multiple equilibria there is no reason to expect that overconfidence will result in a unique equilibrium. For similar reasons there is no reason to expect that in general an increase in risk aversion in the rank dependent model would reduce the multiplicity of equilibria in a stochastic game.

The two sections used different techniques. The results on overconfidence were obtained using monotone comparative statics. However these techniques are less suitable for studying games with ambiguity. To prove comparative statics results it is necessary to assume increasing differences. However a Choquet integral of utility displays increasing differences with respect to an ambiguity parameter if and only if the underlying utility function is concave. Thus no extra generality could be gained by replacing the assumption of concavity with increasing differences.

## 6.2 Related Literature

Compared to our previous work, e.g. Eichberger and Kelsey (2002), this paper makes a number of innovations. Two of the most important are that we allow for optimistic responses to ambiguity and consider other behavioural theories in addition to ambiguity. Much of the innovation in the present paper involves developing techniques for modelling optimistic responses to ambiguity. New definitions of support and measures of ambiguity-attitude were needed. This has enabled us to make a clear distinction between ambiguity and ambiguity attitude. In our previous research Eichberger and Kelsey (2002) both ambiguity and ambiguity-attitude were varied simultaneously in our comparative static exercises. (It is hard to avoid this in a model which assumes ambiguity-aversion.) We also consider a larger class of games since Eichberger and Kelsey (2002) restricted attention to symmetric equilibria of symmetric games, these assumptions have been relaxed apart from in the section on strategic substitutes. Possible applications of these results would include oligopoly models, public goods and environmental models. Applications

of our comparative statics results can be found in Eichberger and Kelsey (2002) and Eichberger, Kelsey, and Schipper (1999).

It would be desirable to extend the theory to extensive form games. This poses some new technical problems. Since players will receive new information during the course of play, it will be necessary to model how information is updated during the course of play. Some possible updating rules are discussed in Eichberger, Grant, and Kelsey (2005).

## APPENDIX

### A ALTERNATIVE NOTIONS OF SUPPORT

In this appendix we discuss some alternative notions of support which have been proposed. We show that most of them coincide for the important case of convex capacities. Dow and Werlang (1994) define the support of a capacity to be a minimal set whose complement has capacity 0.

**Definition A.1** *The DW-support of capacity  $\nu$ ,  $\text{supp}_{DW} \nu$  is a set  $E \subseteq S_{-i}$ , such that  $\nu(S_{-i} \setminus E) = 0$  and  $\nu(F) > 0$ , for all  $F$  such that  $S_{-i} \setminus E \subsetneq F$ .*

This definition has the advantage that  $\text{supp}_{DW} \nu$  always exists, however it may not be unique. In contrast, Marinacci (2000) defines the support of a capacity  $\nu$  to be the set of states with positive capacity.

**Definition A.2** *The M-support of capacity  $\nu$ , is defined by  $\text{supp}_M \nu = \{s \in S_{-i} : \nu(s) > 0\}$ .*

When it exists  $\text{supp}_M \nu$  is always unique. However there are capacities for which it does not exist, for example the *complete uncertainty capacity*, (see definition 2.6).

It is difficult to apply Definitions A.1 and A.2 when decision makers may be ambiguity-loving. For capacities which are not necessarily ambiguity-averse, it is quite possible that *all* states will get positive capacity. As an example consider a neo-additive capacity  $\nu = \delta(1 - \alpha) + (1 - \delta)\pi$ . Then if  $\alpha > 0$ ,  $\text{supp}_{DW} \nu = \text{supp}_M \nu = S$ .

The following results formally demonstrate the relationships between the various support notions.

**Lemma A.1** *If  $\nu$  is a capacity then  $\text{supp}_M \nu \subset \text{supp}_{DW} \nu$ .*

**Proof.** Let  $\tilde{s} \in \text{supp}_M \nu$  and suppose, if possible,  $\tilde{s} \notin \text{supp}_{DW} \nu$ . Then  $\nu(S \setminus \text{supp}_{DW} \nu) \geq \nu(\tilde{s}) > 0$ , which is a contradiction. ■

The following lemma shows that our definition of support coincides with the M-support for a convex capacity.

**Proposition A.1** *If  $\nu$  is an convex capacity with support  $\text{supp} \nu$ , then  $\text{supp} \nu = \text{supp}_M \nu = \mathcal{B}(\nu)$ .*

**Proof.** Let  $\tilde{s} \in \text{supp}_M \nu$  and let  $\pi \in \text{core} \nu$ . Then  $\pi(\tilde{s}) \geq \nu(\{\tilde{s}\}) > 0$ . Hence,  $\text{supp}_M \nu \subseteq \bigcap_{\pi \in \mathcal{C}(\nu)} \text{supp} \pi = \text{supp} \nu$ . On the other hand, suppose  $s \in \bigcap_{\pi \in \mathcal{C}(\nu)} \text{supp} \pi$ . Since  $\nu$  is convex,  $\nu(s) = \min_{\pi \in \mathcal{C}(\nu)} \pi(s) > 0$ .<sup>17</sup> Hence  $\bigcap_{\pi \in \mathcal{C}(\nu)} \text{supp} \pi \subseteq \text{supp}_M \nu$ .

Suppose  $s \in \text{supp}_M \nu$ . Then  $\nu(s) > 0$ . For any  $A \subset S$ ,  $s \notin A$ , by convexity of  $\nu$ ,  $\nu(A \cup s) \geq \nu(A) + \nu(s) > \nu(A)$ . Hence,  $s \in \mathcal{B}(\nu)$ . Conversely suppose  $s \in \mathcal{B}(\nu)$ , then  $\nu(s) = \nu(\emptyset \cup s) > \nu(\emptyset) = 0$ . Hence,  $s \in \text{supp}_M \nu$ . Thus  $\text{supp}_M \nu = \mathcal{B}(\nu)$ . The result follows. ■

Even for a convex capacity, our support notion does not coincide with that of Dow and Werlang. An example of this is the complete uncertainty capacity,  $\nu^0$ , (see definition 2.6), which has empty support but does have a non-unique DW-support.<sup>18</sup>

**Lemma A.2** *Let  $\mu \nu$  be a capacity on  $S_{-i}$  then  $\text{supp}_{DW} \nu$  is unique if and only if  $\text{supp}_M \nu$  is a DW-support.*

**Proof.** Suppose that  $\text{supp}_{DW} \nu$  is unique. Let  $E$  be a DW-support. By Lemma A.1,  $\text{supp}_M \nu \subseteq E$ . Suppose, if possible, there exists  $\hat{s} \in E \setminus \text{supp}_M \nu$ , then  $\nu(\hat{s}) = 0$ . Hence,  $F := S \setminus \{\hat{s}\}$  satisfies  $\nu(S \setminus F) = 0$ . Let  $G$  be a minimal set such that  $G \subseteq F$  and  $\nu(S \setminus G) = 0$ . Then,  $G \neq E$  is another DW-support which contradicts uniqueness. Hence  $\text{supp}_M \nu = \text{supp}_{DW} \nu$ .

Now suppose  $\text{supp}_M \nu$  is a DW-support. Let  $F$  be an arbitrary DW-support. By Lemma A.1,  $\text{supp}_M \nu \subset F$ . Thus by the minimality part of the definition of a DW-support, we must have  $F = \text{supp}_M \nu$ , which gives uniqueness. The result follows. ■

Thus for a convex capacity if the DW-support is unique, the various support notions coincide. The following proposition summarises the above analysis.

**Proposition A.2** *For a convex capacity  $\nu$*

<sup>17</sup> Although  $\text{core} \nu$  is an infinite set, the minimum must occur at one of the extremal points. The set of extremal points of a core is finite. Thus the minimum must be positive.

<sup>18</sup> If  $s \in S$ , then  $\{s\}$  is a DW-support of  $\nu^0$ .

1. if  $\text{supp}_{DW}$  is unique, then  $\mathcal{B}(\mu\nu) = \text{supp}_M \nu = \text{supp} \nu = \text{supp}_{DW} \nu$ ,
2. otherwise,  $\text{supp}_M \nu = \mathcal{B}(\nu) = \text{supp} \nu \subset \text{supp}_{DW} \nu$ .

**Proof.** Part 1 follows from Proposition A.1 and Lemma A.2. Part 2 follows from Proposition A.1 and Lemma A.1. ■

Alternatively the support of a capacity could be defined to be the complement of the set of states which are always given zero weight in the Choquet integral. Solution concepts for games based on this notion of support have been studied by Dow and Werlang (1991) and Lo (1996). They show that this definition of support does not result in a solution concept which is significantly different to Nash equilibrium. These conclusions seem incompatible with our objective of modelling deviations from Nash equilibrium due to ambiguity.

## B PROOFS

This appendix contains the proofs of our main results and some supporting results.

### B.1 Ambiguity

If player  $i$  has beliefs represented by the capacity  $\nu_i^{\tilde{s}} = \nu_i^{\tilde{s}}(\alpha_i, \delta_i, \mu_i)$  then intuitively (s)he believes his/her opponents will play strategy profile  $\tilde{s}$ , however (s)he lacks confidence in this belief. The following lemma says that if a given player believes (in this sense) his/her opponents will play a higher strategy profile then this will raise his/her marginal benefit of increasing his/her own strategy.

**Lemma B.1** *Let  $\Gamma$  be a game with positive aggregate externalities and strategic complements, then if  $\hat{s} \geq \tilde{s}, \forall s_i \in S_i, \text{MB}_i(s_i, \nu^{\hat{s}}) \geq \text{MB}_i(s_i, \nu^{\tilde{s}})$ .*

**Proof.** By Lemma B.3,  $\text{MB}_i(s_i, \nu^{\hat{s}}) - \text{MB}_i(s_i, \nu^{\tilde{s}}) = \delta [\Delta_i(s_i, \hat{s}) - \Delta_i(s_i, \tilde{s})] > 0$ , by strategic complementarity. ■

**Lemma B.2** *Let  $\chi$  be a convex capacity on  $S$  with  $\text{supp} \chi = \emptyset$ . Define a capacity  $\xi$  on  $S$  by  $\xi = \delta\pi + (1 - \delta)\chi$  then:*

1. Core of  $\xi = \{\delta\pi + (1 - \delta)p : p \in \mathcal{C}(\chi)\}$ ;

$$2. \text{ supp } \xi = \text{ supp } \pi;$$

$$3. \tilde{\xi} = \delta\pi(A) + (1 - \delta)\tilde{\chi}(A).$$

**Proof.** If  $p \in \mathcal{C}(\chi)$  then for all  $A \subseteq S$ ,  $p(A) \geq \chi(A)$ , hence  $\delta\pi(A) + (1 - \delta)p(A) \geq \delta\pi(A) + (1 - \delta)\chi(A)$ , which implies  $\delta\pi + (1 - \delta)p \in \mathcal{C}(\chi)$ . Conversely suppose  $q \in \mathcal{C}(\chi)$ . Then for all  $A \subseteq S$ ,  $q \geq \delta\pi(A) + (1 - \delta)\chi(A)$  hence  $\frac{1}{(1 - \delta)}(q - \delta\pi(A)) \geq \chi(A)$ , which implies  $\frac{1}{(1 - \delta)}(q - \delta\pi(A)) \in \mathcal{C}(\chi)$ . Hence  $\mathcal{C}(\xi) = \{\delta\pi + (1 - \delta)p : p \in \mathcal{C}(\chi)\}$ .

It is clear that  $\xi$  is convex since a linear combination of convex capacities is convex.

Suppose that  $\hat{s} \in \text{supp } \pi$ , then  $\delta\pi(\hat{s}) + (1 - \delta)p(\hat{s}) > 0$  for all  $p \in \mathcal{C}(\chi)$ , which implies  $\hat{s} \in \text{supp } \xi$ . Conversely suppose that  $\tilde{s} \in \text{supp } \xi$ . Then since  $\text{supp } \chi = \emptyset$ , there exists  $q \in \mathcal{C}(\chi)$  such that  $q(\tilde{s}) = 0$ . Since  $\tilde{s} \in \text{supp } \xi$ ,  $\delta\pi(\tilde{s}) + (1 - \delta)q(\tilde{s}) > 0$ , which implies  $\pi(\tilde{s}) > 0$  hence  $\tilde{s} \in \text{supp } \pi$ . Thus  $\text{supp } \xi = \text{supp } \pi$ .

Part 3 follows directly from the definitions. ■

**Proof of Theorem 3.1** The first order conditions for a pure equilibrium in which  $\hat{s} = \langle \hat{s}_1, \dots, \hat{s}_n \rangle$  is the equilibrium strategy may be expressed as  $\text{MB}_i(\hat{s}_i, \nu^{\hat{s}}) \geq 0$ ,  $\text{MB}_i(\hat{s}_i + 1, \nu^{\hat{s}}) \leq 0$ , or  $\text{MB}_i(\hat{s}_i, \nu^{\hat{s}}) < 0$ ,  $\hat{s}_i = 0$ .<sup>19</sup>

Let  $\mathcal{T} = \{s \in S : \forall i, \text{ either } \text{MB}_i(s_i, \nu^{\hat{s}}) \geq 0 \text{ or } s_i = 0\}$ . Note that  $\mathcal{T}$  is non-empty, since  $\mathbf{0} \in \mathcal{T}$ . Let  $\tilde{s}$  be a maximal element of  $\mathcal{T}$ . Then

$$\forall i, \text{MB}_i(\tilde{s}_i, \nu^{\tilde{s}}) \geq 0 \text{ or } \tilde{s}_i = 0. \quad (1)$$

Let  $\mathcal{I} = \{i : \tilde{s}_i \neq K_i\}$ , i.e.  $\mathcal{I}$  denotes the set of individuals who are not playing their highest strategy. If  $i \in \mathcal{I}$ , define  $\tilde{s}^{+i} = \langle \tilde{s}_{-i}, \tilde{s}_i + 1 \rangle$ . Then since  $\tilde{s}$  is maximal,  $\exists j, \text{MB}_j(\tilde{s}^{+i}, \nu^{\tilde{s}^{+i}}) < 0$  and  $\tilde{s}_j^{+i} \neq 0$ . However by Lemma B.1,  $\tilde{s}^{+i} \geq \tilde{s}$  implies that if  $j \neq i$  and  $\tilde{s}_j \neq 0$ ,  $\text{MB}_j(\tilde{s}_j, \nu^{\tilde{s}^{+i}}) \geq 0$ . Hence it is that case that  $\forall i \in \mathcal{I}, \text{MB}_i(\tilde{s}_i + 1, \nu^{\tilde{s}^{+i}}) < 0$ . By Lemma B.1,

$$\forall i \in \mathcal{I}, \text{MB}_i(\tilde{s}_i + 1, \nu^{\tilde{s}}) < 0. \quad (2)$$

Equations (1) and (2) imply  $\nu^{\tilde{s}}$  is a pure equilibrium, in which all play  $\tilde{s}$  and beliefs are represented by CC capacities with degrees of ambiguity-aversion  $\langle \alpha_1, \dots, \alpha_n \rangle$ .<sup>20</sup> ■

<sup>19</sup>Recall  $\nu_i^{\tilde{s}} = \nu_i(\pi_i^{\tilde{s}}, \chi_i, \delta_i, \alpha_i)$ , where  $\pi_i^{\tilde{s}}$  denotes the probability distribution on  $S_{-i}$ , which assigns probability 1 to  $\tilde{s}_{-i}$ .

<sup>20</sup>The strategy of proof is similar to that of Tarski's fixed point theorem, see Tarski (1955).

**Theorem 3.2** Let  $\Gamma$  be a symmetric game, then for any given numbers  $\alpha, \delta, 0 \leq \alpha, \delta \leq 1$  and any anonymous capacity  $\chi$ ,  $\Gamma$  has a symmetric equilibrium  $\nu_i = \nu_i(\pi_i, \chi_i, \delta_i, \alpha_i)$  in CC capacities.

**Proof of Theorem 3.2** Define  $\Psi(s_i, s_{-i}) = u(s_i, s_{-i}) + (1 - \delta)\alpha \int u(s_i, s_{-i}) d\chi + (1 - \delta)(1 - \alpha) \int u(s_i, s_{-i}) d\tilde{\chi}$ , for  $1 \leq i \leq n$ . Consider the symmetric game (without ambiguity)  $\Gamma' = \langle N, (S_i)(u_i) : 1 \leq i \leq n \rangle$ , where the players have strategy sets  $\{0, 1, \dots, m^*\}$  and player  $i$ 's utility function is  $\Psi$ . Since the sum of concave functions is concave,  $\Psi$  is concave in  $s_i$ . Hence the game  $\Gamma'$  has a symmetric Nash equilibrium, in which players independently choose a strategy according to the probability distribution  $\pi_i^*$  on  $S_i$ . (see Moulin (1986) p. 115). Let  $\pi_{-i}^*$  denote probability distribution on  $S_{-i}$ , which is the independent product of the marginals,  $\pi_j^*, j \neq i$ .

Define by  $\nu^*(A) = \delta\pi_{-i}^*(A) + (1 - \delta)[\alpha\chi(A) + (1 - \alpha)\tilde{\chi}(A)]$ . We assert that the profile of CC capacities  $\nu^* = \langle \nu_1^*, \dots, \nu_n^* \rangle$  is a symmetric equilibrium of  $\Gamma$ . Consider player  $i$ . Suppose that strategy  $\hat{s}_i$  is in the support of  $\nu^*$ . Then by Lemma 3.2,  $\pi^*(\hat{s}_i) > 0$ . Thus strategy  $\hat{s}_i$  is given positive probability in the Nash equilibrium of the game  $\Gamma'$ . This implies,  $\mathbf{E}_{\pi^*} \Psi(\hat{s}_i, \hat{s}_{-i}) \geq \mathbf{E}_{\pi^*} \Psi(\hat{s}_i, \hat{s}_{-i})$ , for  $\hat{s}_i \in S_i$ , where the expectation is over  $\hat{s}_{-i}$ .

Expanding,  $\int \Psi^i(\hat{s}_i, s_{-i}) d\pi^*(s_{-i}) = \delta \int u(\hat{s}_i, s_{-i}) d\pi^*(s_{-i}) + (1 - \delta)\alpha \int u(\hat{s}_i, s_{-i}) d\chi + (1 - \delta)(1 - \alpha) \int u(\hat{s}_i, s_{-i}) d\tilde{\chi} \geq \delta \int u(s_i, s_{-i}) d\pi^*(s_{-i}) + (1 - \delta)\alpha \int u(s_i, s_{-i}) d\chi + (1 - \delta)(1 - \alpha) \int u(s_i, s_{-i}) d\tilde{\chi}$  for  $s_i \in S_i$ . This is equivalent to  $\int u(\hat{s}_i, s_{-i}) d\nu^*(s_{-i}) \geq \int u(s_i, s_{-i}) d\nu^*(s_{-i})$ , which establishes that  $\hat{s}_i$  is a best response for player  $i$ , given that his/her beliefs can be represented by capacity  $\nu^*$ . It follows that  $\nu^*$  is a symmetric equilibrium with ambiguity. ■

**Notation B.1** For given  $\alpha_i, \delta_i, \chi_i$  and  $\pi_i$ , define  $\varsigma_r(\alpha_i) = \alpha_i[\chi_i(H_r) - \chi_i(H_{r+1})] + (1 - \alpha_i)\chi_i(\neg H_{r+1}) - (1 - \alpha_i)\chi_i(\neg H_r)$ , for  $1 \leq r \leq M - 1$ ;  $\varsigma_M(\alpha_i) = (1 - \alpha_i) + \alpha_i\chi(H_M) - (1 - \alpha_i)\chi_i(\neg H_M)$ ,  $c_r(\alpha_i) = \delta_i[\pi_i(H_r) - \pi_i(H_{r+1})] + (1 - \delta_i)\varsigma_r(\alpha)$ , for  $1 \leq r \leq M - 1$  and  $c_M(\alpha_i) = \delta_i\pi(H_M) + (1 - \delta_i)\varsigma_M(\alpha_i)$ .

Thus the  $c_r$ 's are the decision weights in the Choquet integral with respect to a CC capacity.

**Lemma B.3** Let  $\Gamma$  be a game with positive aggregate externalities and let the beliefs of player



$i$  be represented by the CC capacity  $\nu_i = \delta_i \pi_i + (1 - \delta_i) [\alpha_i \chi + (1 - \alpha_i) \tilde{\chi}_i]$  on  $S_{-i}$ , then

$$\begin{aligned} \text{MB}_i(s_i, \nu_i) &= \sum_{r=0}^M \Delta_i(s_i, f_i^r) c_r(\alpha_i) \\ &= \delta_i \pi(H_M) \Delta_i(s_i, f_M) + \sum_{r=0}^M \Delta_i(s_i, f_i^r) \delta_i [\pi_i(H_r) - \pi_i(H_{r+1})] + (1 - \delta_i) \sum_{r=0}^M \Delta_i(s_i, f_i^r) \varsigma_r(\alpha_i). \end{aligned} \quad (3)$$

**Proof.** Consider player  $i$ . Then  $\int u_i(s_i, s_{-i}) d\nu_i(s_{-i}) = u_i(s_i, f_M) \nu_i(H_M) + \sum_{r=0}^{M-1} u_i(s_i, f_r) [\nu_i(H_r) - \nu_i(H_{r+1})]$ . Similarly  $\int u_i(s_i - 1, s_{-i}) d\nu_i(s_{-i}) = u_i(s_i - 1, f_M) \nu_i(H_M) + \sum_{r=0}^{M-1} u_i(s_i - 1, f_r) [\nu_i(H_r) - \nu_i(H_{r+1})]$ . By taking the difference of these two expressions we obtain:  $\text{MB}_i(s_i, \nu_i) = \Delta_i(s_i, f_M) \nu_i(H_M) + \sum_{r=0}^{M-1} \Delta_i(s_i, f_r) [\nu_i(H_r) - \nu_i(H_{r+1})]$ . If we substitute  $\nu_i(H_r) = \delta_i \pi_i(H_r) + (1 - \delta_i) \alpha_i \chi_i(H_r) + (1 - \delta_i) (1 - \alpha_i) \tilde{\chi}_i(H_r)$ , we obtain  $\text{MB}_i(s_i, \nu_i) = \Delta_i(s_i, f_M) \delta_i \pi_i(H_M) + \Delta_i(s_i, f_M) (1 - \delta_i) \{[\alpha_i \chi_i(H_M) + (1 - \alpha_i) \tilde{\chi}_i(H_M)]\} + \sum_{r=0}^{M-1} \Delta_i(s_i, f_r) \delta_i [\pi_i(H_r) - \pi_i(H_{r+1})] + \sum_{r=0}^{M-1} \Delta_i(s_i, f_r) (1 - \delta_i) [\alpha_i \chi_i(H_r) + (1 - \alpha_i) \tilde{\chi}_i(H_r)] - \sum_{r=0}^{M-1} \Delta_i(s_i, f_r) (1 - \delta_i) [\alpha_i \chi_i(H_{r+1}) + (1 - \alpha_i) \tilde{\chi}_i(H_{r+1})]$   
 $= \Delta_i(s_i, f_M) \{ \delta_i \pi_i(H_M) + (1 - \delta_i) [\alpha_i \chi_i(H_M) + (1 - \alpha_i) - (1 - \alpha_i) \chi_i(\neg H_M)] \}$   
 $+ \sum_{r=0}^{M-1} \Delta_i(s_i, f_r) \{ \delta_i [\pi_i(H_r) - \pi_i(H_{r+1})] + (1 - \delta_i) [\alpha_i \chi_i(H_r) + (1 - \alpha_i) [1 - \chi_i(\neg H_r)]] - (1 - \delta_i) [\alpha_i \chi_i(H_{r+1}) + (1 - \alpha_i) [1 - \chi_i(\neg H_{r+1})]] \}.$   
Hence  $\text{MB}_i(s_i, \nu_i) = \Delta_i(s_i, f_M) \{ \delta_i \pi_i(H_M) + (1 - \delta_i) \varsigma_M(\alpha_i) \} + \sum_{r=0}^{M-1} \Delta_i(s_i, f_r) \{ \delta_i [\pi(H_r) - \pi(H_{r+1})] + (1 - \delta_i) \varsigma_r(\alpha_i) \}$  from which, the result follows. ■

**Lemma B.4** Let  $\Gamma$  be a game with positive aggregate externalities. Then for any given  $n$ -tuples  $\alpha, \delta$  and  $\chi$ , suppose that  $\text{MB}_i(\tilde{s}_i, \nu_i^{\tilde{s}}) \geq 0$  for  $1 \leq i \leq n$ .<sup>21</sup> Then there exists  $\hat{s} \geq \bar{s}$  such that there is a pure equilibrium in CC capacities  $\nu^* = \nu^{\hat{s}} = \nu(\pi^{\hat{s}}, \chi, \delta, \alpha)$  in which  $\hat{s}$  is the equilibrium strategy.

**Proof.** Suppose first that  $\text{MB}_i(s'_i, \nu_i^{s'_i}) \geq 0$ , for all  $s' \geq \bar{s}$ , then there is a corner equilibrium, in which player  $i$  plays  $K_i$ , for  $1 \leq i \leq n$ . Otherwise, let  $\mathcal{K} = \{s \in S : s \geq \bar{s}, \forall i, \text{MB}_i(s'_i, \nu_i^{s'_i}) \geq 0\}$ . Let  $\tilde{k}$  be a maximal element of  $\mathcal{K}$ , (maximal elements exist since the space of all strategy profiles is finite).

<sup>21</sup>Recall if,  $\tilde{s} \in S$  is a given strategy profile  $\pi_i^{\tilde{s}}$  is the probability distribution on  $S_{-i}$ , which assigns probability 1 to  $\tilde{s}_{-i}$  then  $\nu_i^{\tilde{s}} = \nu_i^{\tilde{s}}(\alpha_i, \delta_i, \chi_i) = \delta_i \pi_i^{\tilde{s}} + (1 - \delta_i) [\alpha_i \chi_i + (1 - \alpha_i) \tilde{\chi}_i]$ .

Then we may show that  $\tilde{k}$  is an equilibrium strategy profile when beliefs are  $\nu^{\hat{k}}$ . The proof can be completed in a similar way to the last part of the proof of Theorem 3.1. ■

The next result shows that if individual  $i$ 's ambiguity-aversion increases then his/her perceived marginal benefit falls.

**Lemma B.5** *Let  $\Gamma$  be a game with positive aggregate externalities and strategic complements.*

*Assume  $\alpha_i^* \geq \alpha'_i$ , then  $\text{MB}_i(s_i, \nu_i^{\tilde{s}}(\alpha'_i, \delta_i, \chi_i)) > \text{MB}_i(s_i, \nu_i^{\tilde{s}}(\alpha_i^*, \delta_i, \chi_i))$ .*

**Proof.** Note that  $\sum_{r=t}^M \varsigma_r(\alpha) = [\alpha \chi_i(H_M) - (1 - \alpha) \chi_i(\neg H_M)] + \sum_{r=t}^{M-1} [\alpha \chi_i(H_r) - \alpha \chi_i(H_{r+1})] + \sum_{r=t}^{M-1} (1 - \alpha) [\chi_i(\neg H_{r+1}) - \chi_i(\neg H_r)] = (1 - \alpha) + \alpha \chi_i(H_t) - (1 - \alpha) \chi_i(\neg H_t)$ . Hence

$$\sum_{r=t}^M \varsigma_r(\alpha) = \alpha [\chi_i(H_t) + \chi_i(\neg H_t) - 1] + 1 - \chi_i(\neg H_t). \quad (4)$$

Since  $\chi_i$  is strictly sub-additive,  $\sum_{r=t}^M \varsigma_r(\alpha'_i) > \sum_{r=t}^M \varsigma_r(\alpha_i^*)$ , hence the  $\varsigma_r(\alpha'_i)$ 's first order stochastically dominate the  $\varsigma_r(\alpha_i^*)$ 's. Since  $\Delta_i(s, f_r)$  is strictly increasing in  $r$ ,

$$\sum_{r=0}^M \varsigma_r(\alpha'_i) \Delta_i(s, f_i^r) < \sum_{r=0}^M \varsigma_r(\alpha_i^*) \Delta_i(s, f_i^r). \quad (5)$$

The result follows from equation (3). ■

**Lemma B.6** *Let  $\Gamma$  be a game with positive aggregate externalities. Consider a given player,  $i$  say. Let  $m_1 > m_2 > m_3$ , be three possible strategies for player  $i$ . Suppose that with beliefs over  $S_{-i}$  given by  $\nu$ ,  $m_1$  is indifferent to  $m_3$ , then  $m_2$  is strictly preferred to  $m_1$ .*

**Proof.** Playing strategy  $m_i$  yields (Choquet) expected utility:  $V_i(m_i) = \int u(m_i, s_{-i}) d\nu(s_{-i})$ . There exists  $\lambda$  such that  $m_2 = \lambda m_1 + (1 - \lambda) m_3$ . Since  $m_1$  and  $m_3$  are indifferent:  $V_i(m_1) = \lambda V_i(m_1) + (1 - \lambda) V_i(m_3) = \lambda \int u_i(m_1, s_{-i}) d\nu(s_{-i}) + (1 - \lambda) \int u_i(m_3, s_{-i}) d\nu(s_{-i}) = \int [\lambda u_i(m_1, s_{-i}) + (1 - \lambda) u_i(m_3, s_{-i})] d\nu(s_{-i})^{22} < \int u_i(m_2, s_{-i}) d\nu(s_{-i}) = V(m_2)$ , since  $u_i$  is strictly concave in  $s_i$ . The result follows. ■

**Lemma B.7** *Consider a game of positive aggregate externalities with strategic complements.*

*There exist  $\bar{\alpha}$  and  $\bar{\lambda}$  such that if the minimal degree of ambiguity is  $\lambda(\mu_i) \geq \bar{\lambda}$  and  $\alpha_i \geq \bar{\alpha}$ ,*

---

<sup>22</sup>This step is valid since, if there are positive aggregate externalities  $u(m_1, s_{-i})$  and  $u(m_3, s_{-i})$  are comonotonic.

(resp.  $\leq \underline{\alpha}$ ) then in any equilibrium  $\nu_i = \alpha_i \mu_i + \tilde{\alpha}_i \tilde{\mu}_i$ ,  $\text{supp } \nu_i \subseteq \text{argmax}_{s_i \in S_i} u_i(s_i, \bar{s}_{-i})$ , (resp.  $\text{supp } \nu_i \subseteq \text{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i})$ ).

**Proof.** Let  $s_i'' \in \text{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i})$  and let  $\epsilon = u_i(s_i'', \bar{s}_{-i}) - \max_{s_i \notin \text{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i})} u_i(s_i, \underline{s}_{-i})$ .

By construction  $\epsilon > 0$ . Let  $\hat{\nu}_i = \alpha_i \mu_i + \tilde{\alpha}_i \tilde{\mu}_i$  denote  $i$ 's equilibrium beliefs. If individual  $i$  plays

strategy  $s_i''$  (s)he receives utility:  $u_i(s_i'', f_M) \nu_i(H_M) + \sum_{r=0}^{M-1} u_i(s_i'', f_r) [\nu_i(H_r) - \nu_i(H_{r+1})]$

$$= u_i(s_i'', f_M) (\alpha_i \mu_i(H_M) + \tilde{\alpha}_i \tilde{\mu}_i(H_M)) + \sum_{r=0}^{M-1} u_i(s_i'', f_r) \alpha_i (\mu_i(H_r) - \mu_i(H_{r+1})) \\ + \sum_{r=0}^{M-1} u_i(s_i'', f_r) \tilde{\alpha}_i [\tilde{\mu}_i(H_r) - \tilde{\mu}_i(H_{r+1})].$$

Since  $\tilde{\alpha}_i \rightarrow 0$  as  $\alpha_i \rightarrow 1$ , the terms involving  $\tilde{\alpha}_i$  can be neglected, hence this is approximately,  $u_i(s_i'', f_M) \alpha_i \mu_i(H_M) + \alpha_i \sum_{r=0}^{M-1} u_i(s_i'', f_r) (\mu_i(H_r) - \mu_i(H_{r+1})) = u_i(s_i'', f_M) \alpha_i \mu_i(H_M) + \sum_{r=0}^{M-1} u_i(s_i'', f_r) \alpha_i (\mu_i(H_r) - \mu_i(H_{r+1}))$ .

Similarly, if individual  $i$  plays strategy  $s_i' \notin \text{argmax}_{s_i \in S_i} u_i(s_i, \bar{s}_{-i})$  his her utility is approximately:

$$V(s_i') = u_i(s_i', f_M) \alpha_i \mu_i(H_M) + \sum_{r=0}^{M-1} u_i(s_i', f_r) \alpha_i (\mu_i(H_r) - \mu_i(H_{r+1})).$$

If we define  $d_r = u_i(s_i'', f_r) - u_i(s_i', f_r)$ , the extra utility  $i$  gets from playing strategy  $s_i''$  rather than  $s_i'$ , is approximately:  $V(s_i'') - V(s_i') = d_M \alpha_i \mu_i(H_M) + \sum_{r=1}^{M-1} d_r \alpha_i (\mu_i(H_r) - \mu_i(H_{r+1}))$

$$= d_0 (\mu_i(H_0) - \mu_i(H_1)) + \sum_{r=1}^{M-1} d_r \alpha_i (\mu_i(H_r) - \mu_i(H_{r+1})) + d_M \alpha_i \mu_i(H_M) \\ = d_0 (1 - \mu_i(H_1)) + \sum_{r=1}^{M-1} d_r \alpha_i (\mu_i(H_r) - \mu_i(H_{r+1})) + d_M \alpha_i \mu_i(H_M) \\ \geq d_0 (1 - \mu_i(H_1)) + d_M \mu_i(H_1) \geq \bar{\lambda} d_0 + d_M (1 - \bar{\lambda}) \geq \bar{\lambda} \epsilon + d_M (1 - \bar{\lambda}) > 0, \text{ provided } \bar{\lambda} \text{ is sufficiently large.}$$

The first inequality follows since strategic complementarity implies that  $d_r$  is increasing in  $r$ . It follows that for sufficiently large minimal degrees of ambiguity and large  $\alpha_i$ , individual  $i$  will play a strategy from  $\text{argmax}_{s_i \in S_i} u_i(s_i, \underline{s}_{-i})$  in equilibrium. ■

**Notation B.2** If  $1 \leq k \leq m^*$ , we shall use  $r(k)$  to denote that value of  $r$  which satisfies  $f_{r(k)} = f(k, \dots, k)$ .

**Proposition 4.2** Consider a game of *positive* aggregate externalities with *strategic substitutes*. Let  $\bar{\nu} = \nu(\pi, \chi, \delta, \bar{\alpha})$  (resp.  $\hat{\nu} = \nu(\pi, \chi, \delta, \hat{\alpha})$ ) be a symmetric equilibrium in CC-capacities in which  $\bar{k}$  (resp.  $\hat{k}$ ) is the highest strategy played and  $\bar{\ell}$  (resp.  $\hat{\ell}$ ) is the lowest strategy played. Then  $\hat{\alpha} > \bar{\alpha}$  implies  $\hat{k} \geq \bar{k}$  and  $\hat{\ell} \leq \bar{\ell}$ .

**Proof of Proposition 4.2** Suppose if possible  $\hat{k} < \bar{k}$ , we shall show that this leads to a contradiction.

**Claim** If we assume  $\hat{k} < \bar{k}$ , then  $\bar{\pi}(H_t) \geq \hat{\pi}(H_t)$  for  $1 \leq t \leq M$ .

Consider first the case where  $\hat{\nu}$  is a pure equilibrium, then  $\hat{\pi}(H_t) = 0$ , if  $r(\hat{k}) > t$ ,  $= 1$  otherwise. Similarly  $\bar{\pi}(H_t) = 0$ , if  $t > r(\bar{k})$  and  $\bar{\pi}(H_t) = 1$  if  $r(\bar{k} - 1) \geq t$ . Since  $\hat{k} < \bar{k}$ ,  $\bar{k} - 1 \geq \hat{k}$ . Thus if  $t > r(\bar{k} - 1)$ ,  $\bar{\pi}(H_t) \geq \hat{\pi}(H_t) = 0$ ; for  $r(\bar{k} - 1) \geq t$ ,  $1 = \bar{\pi}(H_t) \geq \hat{\pi}(H_t)$ , which establishes the claim in this case.

Now consider the case where  $\hat{\nu}$  is a mixed equilibrium. By Lemma B.6 the equilibrium strategies must be  $\hat{k}, \hat{k} - 1$ . Note that  $\hat{\pi}(H_t) = 0$ , if  $t > r(\hat{k})$ ,  $= 1$  if  $r(\hat{k} - 1) \leq t$ . Similarly  $\bar{\pi}(H_t) = 0$ , if  $t > r(\bar{k})$  and  $\bar{\pi}(H_t) = 1$  if  $r(\bar{k} - 1) \geq t$ . Since  $\hat{k} < \bar{k}$ ,  $\bar{k} - 1 \geq \hat{k}$ . Thus for  $t > r(\hat{k})$ ,  $\bar{\pi}(H_t) \geq \hat{\pi}(H_t) = 0$ ; if  $r(\bar{k} - 1) \geq t$ , if  $\bar{\pi}(H_t) = 1 \geq \hat{\pi}(H_t)$ ; which establishes the claim.

By Lemma B.3, we may write  $MB(\bar{\nu}, k) = \sum_{r=0}^M \Delta(s_i, f_r) c_r(\bar{\alpha})$  and  $MB(\hat{\nu}, k) = \sum_{r=0}^M \Delta(s_i, f_r) c_r(\hat{\alpha})$ . Since  $\chi$  is convex,  $\chi(H_t) + \chi(-H_t) \leq 1$ , hence as  $\hat{\alpha} > \bar{\alpha}$ ,  $\hat{\alpha}[\chi(H_t) + \chi(-H_t) - 1] \leq \bar{\alpha}[\chi(H_t) + \chi(-H_t) - 1]$ . By equation (4),  $\sum_{r=t}^M c_r(\alpha) = \delta \pi(H_M) + \sum_{r=t}^{M-1} \delta [\pi(H_r) - \pi(H_{r+1})] + (1 - \delta) \sum_{r=t}^M c_r(\alpha) = \delta \pi(H_t) + (1 - \delta) \{ \alpha [\chi_i(H_t) + \chi_i(-H_t) - 1] + 1 - \chi_i(-H_t) \}$ . Hence the  $c_r(\bar{\alpha})$ 's first order stochastically dominate the  $c_r(\hat{\alpha})$ 's. Since  $\Delta(s_i, f_r)$  is strictly decreasing in  $r$ , first order stochastic dominance implies  $MB(\bar{\nu}, k) < MB(\hat{\nu}, k)$  for all  $k$ .

A necessary condition for  $\hat{k}$  to be an equilibrium action with beliefs  $\hat{\nu}$  is  $0 \geq MB(\hat{k} - 1, \hat{\nu})$ . Hence by concavity  $0 \geq MB(\bar{k}, \hat{\nu}) > MB(\bar{k}, \bar{\nu})$ . However, a necessary condition for  $\bar{k}$  (resp.  $\bar{k}, \bar{k} + 1$ ) to be an equilibrium contribution level (resp. levels) with beliefs  $\bar{\nu}$ , is  $MB(\bar{k}, \bar{\nu}) \geq 0$ , (resp.  $MB(\bar{k}, \bar{\nu}) = MB(\bar{k} + 1, \bar{\nu}) = 0$ ). This contradiction establishes that  $\hat{k} \leq \bar{k}$ . A similar argument applies to the lowest equilibrium strategy. ■

## B.2 Overconfidence

**Lemma B.8** The function  $\psi^i : S \times \Theta \rightarrow \mathbb{R}$  defined by  $\psi^i(s_i, s_{-i}, \pi^i) = \sum_{j=k}^J \pi_j^i u^i(s_i, s_{-i}, \theta_j)$  has increasing differences in  $s_i$  and  $\pi^i$ .

**Proof.** Let  $\hat{\pi}^i$  be a distribution which first order stochastically dominates  $\tilde{\pi}^i$ . We are required to show that  $\sum_{j=k}^J \hat{\pi}_j^i u^i(s_i, s_{-i}, \theta_j) - \sum_{j=k}^J \tilde{\pi}_j^i u^i(s_i, s_{-i}, \theta_j)$  is an increasing function of  $s_i$ . Let  $k$  denote the number of terms in which  $\hat{\pi}$  and  $\tilde{\pi}$  differ. We shall proceed by induction on  $k$ .

**Case  $k = 2$**  There exist  $j', j''$  with  $j'' > j'$  such that  $\hat{\pi}_{j''}^i > \tilde{\pi}_{j''}^i$ ,  $\hat{\pi}_{j'}^i < \tilde{\pi}_{j'}^i$ , and  $\hat{\pi}_j^i = \tilde{\pi}_j^i$ ,  $j \neq j', j''$ . Then  $\sum_{j=k}^J \hat{\pi}_j^i u^i(s_i, s_{-i}, \theta_j) - \sum_{j=k}^J \tilde{\pi}_j^i u^i(s_i, s_{-i}, \theta_j) = (\hat{\pi}_{j''}^i - \tilde{\pi}_{j''}^i) u^i(s_i, s_{-i}, \theta_{j''}) +$

$(\hat{\pi}_j^i - \tilde{\pi}_j^i) u^i(s_i, s_{-i}, \theta_{j'})$   
 $= (\hat{\pi}_{j''}^i - \tilde{\pi}_{j''}^i) (u^i(s_i, s_{-i}, \theta_{j''}) - u^i(s_i, s_{-i}, \theta_{j'}))$ , which is an increasing function of  $s_i$  since  $u^i(s_i, s_{-i}, \theta)$ , has strictly increasing differences in  $s_i$  and  $\theta$ .

**Inductive step** Assume the result is true whenever  $k \leq K - 1$ . Let  $\hat{\pi}^i$  be a distribution which first order stochastically dominates  $\tilde{\pi}^i$  and differs from it in  $K$  components. Let  $\ell'' = \max \{j : \hat{\pi}_j^i \neq \tilde{\pi}_j^i\}$ , since  $\hat{\pi}$  stochastically dominates  $\tilde{\pi}$ ,  $\hat{\pi}_{\ell''}^i > \tilde{\pi}_{\ell''}^i$ . Let  $\ell' = \max \{j : \hat{\pi}_j^i < \tilde{\pi}_j^i\}$ . Let  $\delta = \min \{\hat{\pi}_{\ell''}^i - \tilde{\pi}_{\ell''}^i, \tilde{\pi}_{\ell'}^i - \hat{\pi}_{\ell'}^i\}$ . Define  $\bar{\pi}^i$  by  $\bar{\pi}_j^i = \hat{\pi}_j^i$ ,  $j \neq \ell', \ell''$ ,  $\bar{\pi}_{\ell''}^i = \hat{\pi}_{\ell''}^i - \delta$ ,  $\bar{\pi}_{\ell'}^i = \hat{\pi}_{\ell'}^i + \delta$ . Then  $\bar{\pi}^i$  differs from  $\tilde{\pi}^i$  in at most  $K - 1$  components and differs from  $\hat{\pi}^i$  in 2 components. By the inductive hypothesis  $\sum_{j=k}^J \hat{\pi}_j^i u^i(s_i, s_{-i}, \theta_j) - \sum_{j=k}^J \bar{\pi}_j^i u^i(s_i, s_{-i}, \theta_j)$  and  $\sum_{j=k}^J \bar{\pi}_j^i u^i(s_i, s_{-i}, \theta_j) - \sum_{j=k}^J \tilde{\pi}_j^i u^i(s_i, s_{-i}, \theta_j)$  are increasing functions of  $s_i$ . The result follows. ■

## References

- BAKER, L., AND R. EMERY (1993): “When Every Relationship is Above Average,” *Law and Human Behaviour*.
- CAMERER, C., AND M. WEBER (1992): “Recent Developments in Modelling Preferences: Uncertainty and Ambiguity,” *Journal of Risk and Uncertainty*, 5, 325–370.
- CAMERER, C. F., G. LOWENSTEIN, AND D. PRELEC (2005): “Neuroeconomics: Why Economics Needs Brains,” *Scandinavian Journal of Economics*, forthcoming.
- CHATEAUNEUF, A., J. EICHBERGER, AND S. GRANT (2005): “Choice under Uncertainty with the Best and Worst in Mind: NEO-Additive Capacities,” *working paper, Universitat Heidelberg*.
- COHEN, M., J.-Y. JAFFRAY, AND T. SAID (1985): “Individual Behaviour under Risk and Uncertainty: An Experimental Study,” *Theory and Decision*, 18, 203–228.
- COMPTE, O., AND A. POSTLEWAITE (2001): “Confidence Enhanced Performance,” *working paper, University of Pennsylvania*.
- DE MEZA, D. (2004): “Overlending,” *working paper, London School of Economics*.
- DOW, J., AND S. R. C. WERLANG (1991): “Nash Equilibrium under Knightian Uncertainty: Breaking Down Backward Induction,” *working paper, London Business School*.

- (1994): “Nash Equilibrium under Uncertainty: Breaking Down Backward Induction,” *Journal of Economic Theory*, 64, 305–324.
- EICHBERGER, J., S. GRANT, AND D. KELSEY (2005): “Updating Choquet Expected Utility Preferences,” *Mathematical Social Sciences*, 49, 143–151.
- EICHBERGER, J., AND D. KELSEY (2000): “Non-Additive Beliefs and Strategic Equilibria,” *Games and Economic Behavior*, 30, 183–215.
- (2002): “Strategic Complements, Substitutes and Ambiguity: The Implications for Public Goods,” *Journal of Economic Theory*, 106, 436–466.
- EICHBERGER, J., D. KELSEY, AND B. SCHIPPER (1999): “Ambiguity and Social Interaction,” *University of Birmingham discussion paper no. 99-26*.
- EPSTEIN, L. G. (1999): “A Definition of Uncertainty Aversion,” *Review of Economic Studies*, 66, 579–606.
- GERVAIS, S., AND I. GOLDSTEIN (2004): “The Effects of Biased Self-Perceptions in Teams,” *working paper Duke University*.
- GERVAIS, S., J. HEATON, AND T. ODEAN (2000): “Capital Budgetting in the Presence of Managerial Overconfidence and Optimism,” *working paper, University of Pennsylvania*.
- GHIRARDATO, P. (1997): “On Independence for Non-Additive Measures and a Fubini Theorem,” *Journal of Economic Theory*, 73, 261–291.
- GHIRARDATO, P., AND M. MARINACCI (2002): “Ambiguity Made Precise: A Comparative Foundation,” *Journal of Economic Theory*, 102, 251–289.
- GILBOA, I. (1989): “Duality in Non-Additive Expected Utility Theory,” *Annals of Operations Research*, 19, 405–414.
- GILBOA, I., AND D. SCHMEIDLER (1989): “Maxmin Expected Utility with a Non-Unique Prior,” *Journal of Mathematical Economics*, 18, 141–153.
- GONZALEZ, R., AND G. WU (1999): “On the Shape of the Probability Weighting Function,” *Cognitive Psychology*, 38, 129–166.

- JAFFRAY, J.-Y., AND F. PHILIPPE (1997): “On the Existence of Subjective Upper and Lower Probabilities,” *Mathematics of Operations Research*, 22, 165–185.
- KELSEY, D., AND S. NANDEIBAM (1996): “On the Measurement of Uncertainty Aversion,” *Discussion paper, The University of Birmingham*.
- KELSEY, D., AND W. SPANJERS (2004): “Ambiguity in Partnerships,” *Economic Journal*, 114, 528–546.
- LO, K. C. (1996): “Equilibrium in Beliefs under Uncertainty,” *Journal of Economic Theory*, 71, 443–484.
- LO, K. C. (1999): “Nash equilibrium without mutual knowledge of rationality,” *Economic Theory*, 14, 621–633.
- MARINACCI, M. (2000): “Ambiguous Games,” *Games and Economic Behavior*, 31, 191–219.
- MILGROM, P., AND J. ROBERTS (1990): “Rationalizability, Learning, and Equilibrium in Games with Strategic Complementarities,” *Econometrica*, 58, 1255–1277.
- MILGROM, P., AND J. ROBERTS (1994): “Comparing Equilibria,” *American Economic Review*, 84, 441–459.
- MILGROM, P., AND C. SHANNON (1994): “Monotone Comparative Statics,” *Econometrica*, 62, 157–180.
- MOULIN, H. (1986): *Game Theory for the Social Sciences*. New York University Press, New York.
- QUIGGIN, J. (1982): “A Theory of Anticipated Utility,” *Journal of Economic Behavior and Organization*, 3, 323–334.
- QUIGGIN, J. (1991): “On The Optimal Design of Lotteries,” *Economica*, 58, 1–16.
- RYAN, M. J. (1997): “CEU Preferences and Game Theoretic Equilibrium,” *The University of Auckland, working paper series no. 167*.
- SARIN, R., AND P. WAKKER (1992): “A Simple Axiomatization of Non-Additive Expected Utility,” *Econometrica*, 60, 1255–1272.

- (1998): “Revealed Likelihood and Knightian Uncertainty,” *Journal of Risk and Uncertainty*, 16, 223–250.
- SCHMEIDLER, D. (1989): “Subjective Probability and Expected Utility without Additivity,” *Econometrica*, 57, 571–587.
- SEGAL, U., AND A. SPIVAK (1990): “First-Order versus Second-Order Risk Aversion,” *Journal of Economic Theory*, 51, 111–125.
- SVENSON, O. (1981): “Are We All Less Risky and More Skillful Than our Fellow Drivers?,” *Acta Psychologica*, 143, 145–46.
- TARSKI, A. (1955): “A Lattice-Theoretical Fixpoint Theorem and its Applications,” *Pacific Journal of Mathematics*, 5, 286–309.
- TOPKIS, D. M. (1998): *Supermodularity and Complementarity*. Princeton University Press, New Jersey.
- VAN DEN STEEN, E. (2002): “Skill or Luck? Biases of Rational Agents,” *MIT working paper*.
- VIVES, X. (1999): *Oligopoly Pricing*. MIT Press, Cambridge, Mass.
- WAKKER, P. (1994): “Separating Marginal Utility and Probabilistic Risk Aversion,” *Theory and Decision*, 36, 1–44.
- WAKKER, P. (2001): “Testing and Characterizing Properties of Nonadditive Measures Through Violations of the Sure Thing Principle,” *Econometrica*, 69, 1039–1060.
- YAARI, M. E. (1987): “Dual Theory of Choice under Uncertainty,” *Econometrica*, 55, 95–115.